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Analyzing and Generating Mathematical Models: An Algebra II Cognitive Tutor Design Study

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Abstract. This paper reports a formative analysis of a Math Modeling Tool in the Algebra II Cognitive Tutor. This tutor is designed to support algebraic reasoning about real world problems. This study focuses on reasoning about situations that can be modeled with general linear form expressions (c = ax + by). Formative evaluations of an early general linear form lesson showed that it helped students comprehend the underlying problem situations, but was less successful in helping students construct symbolic models of the situations. These evaluations guided design of a new tool to scaffold students' understanding of the componential structure of these symbolic models and the mapping of model components to the problem situation. An empirical evaluation shows that the new tool successfully helps students understand the structure of these mathematical models and learn to construct them.

1 Introduction

This paper reports an empirical evaluation of a Math Modeling Tool recently introduced into the Algebra II Cognitive Tutor [3]. The tool is intended to help students learn to write general linear form models (ax + by = c) of problem situations. The tool design was guided by formative evaluations of an initial general linear form lesson and provides students the opportunity to both analyze and write such symbolic models. In the following sections we describe the Algebra II Cognitive Tutor and the problem solving activities that introduce general linear form models. We briefly describe two formative evaluations of these activities and the consequent design of the Math Modeling Tool. We conclude with an empirical evaluation demonstrating the effectiveness of the new tool and a brief discussion of design issues.

2 The Algebra II Cognitive Tutor

The Algebra II Cognitive Tutor builds on a successful collaboration between the Pittsburgh Urban Mathematics Project in the Pittsburgh School District and the Pittsburgh Advanced Cognitive Tutor (PACT) Center at Carnegie Mellon. This collaboration previously yielded the Algebra I Cognitive Tutor [4], now in use in about 150 schools in the United States. The Algebra II Tutor reflects the National

Council of Teachers of Mathematics [5] curriculum standards and its design is guided by three goals: (1) to support students in applying algebra to real-world problems, (2) to support reasoning among multiple representations, (tables, graphs, symbolic expressions, and natural language) and (3) to employ modern computational tools.

This study focuses on problems that can be modeled by general linear form equations, $c = ax + by$. This lesson presents problem statements such as:

> You are selling ads for the school yearbook. There are two types of ads, full-page and half-page. Full-page ads sell for $400 each and half-page ads sell for $250 each. We want to sell enough ads for a total income of $15,000.

The dual lesson objectives are to help students learn to answer numerical questions about these situations, such as: "If you sell 20 full-page ads, how many half-page ads do you need to sell?" and to construct a general linear form model of the situation, e.g., $250x + 400y = 15,000$. Such general linear form relationships pose an interesting challenge because they are an early stumbling block for students in the Algebra II curriculum and because they ultimately form the basis for linear programming.

Fig. 1 displays two windows from a general linear form tutor problem. The problem statement in the upper window describes a problem situation and asks five questions about the situation. The worksheet in the lower window is blank at the beginning of the problem (except for row labels on the left side) and students answer the questions by completing the worksheet. Students (1) identify relevant quantities in the problem and label the columns accordingly; (2) enter appropriate units in the first row of the worksheet; (3) enter a symbolic formula for each quantity in the second row; and (4) answer the questions in the successive table rows. In Fig. 1 the student has completed approximately 3/4 of the problem.

The problem situation displayed in Fig. 1 can be represented as an operator hierarchy, as depicted in Fig. 2. In both the equation and the hierarchical representation in this figure, the variable x represents the number of half-page ads that are sold, the product $250x$ represents the income from half-page ad sales, the variable y represents the number of full-page ads that are sold and the product $400y$ represents the full-page ad sales income. The sum $250x + 400y$ represents total income and is set to the constant $15,000$ in this situation.

These worksheet problems are intended to help students understand the underlying hierarchical structure of the problem situation and to learn to construct general linear form math models. Each question in Fig. 1 is formed by assigning a given value to either a variable or product node in the hierarchy and asking students to compute the value of another variable or product node. For example, Question 2 in Fig. 1 asks "If the income from full-page ads [$400y$] is $4000, how many half-page ads [x] do we need to sell?" Students can answer the questions by traversing the operator hierarchy and successively applying the operators (in moving up through the hierarchy) or the operator inverses (in moving down through the hierarchy). The worksheet formula row requires students to write a symbolic model of the situation. In specifying the formula for "Total Income" in Fig. 1, the student is constructing one side of the general linear form relationship $250x+400y$. The other four formula cells in the worksheet essentially scaffold construction of this expression from its hierarchical constituents, x, y, $250x$ and $400y$. The five question values in the “Total Income” column implicitly encode that this algebraic sum is equal to the constant 15,000.

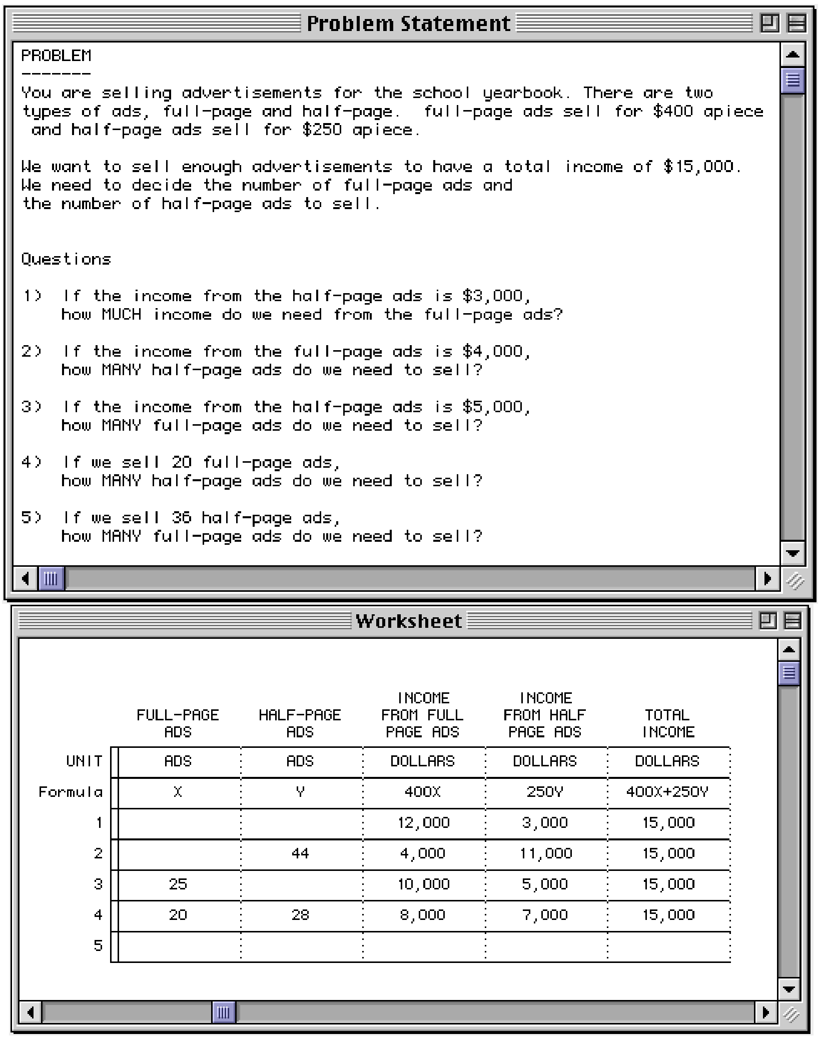

Fig. 1. The PACT Algebra II Tutor interface: A general linear form problem statement and worksheet.

The Algebra II Tutor is constructed around a cognitive model of the problem solving knowledge students are acquiring. The model reflects the ACT-R theory of skill knowledge [1] in assuming that problem solving skills can be modeled as a set of independent production rules. The cognitive model enables the tutor to trace the student's solution path through a complex problem solving space, providing feedback on each problem solving action and advice on problem solving as needed. This *model tracing* process ensures that students reach a successful conclusion to each problem and has been shown to speed learning by as much as a factor of three and to increase achievement levels, in comparison to students solving problems on their own [2].

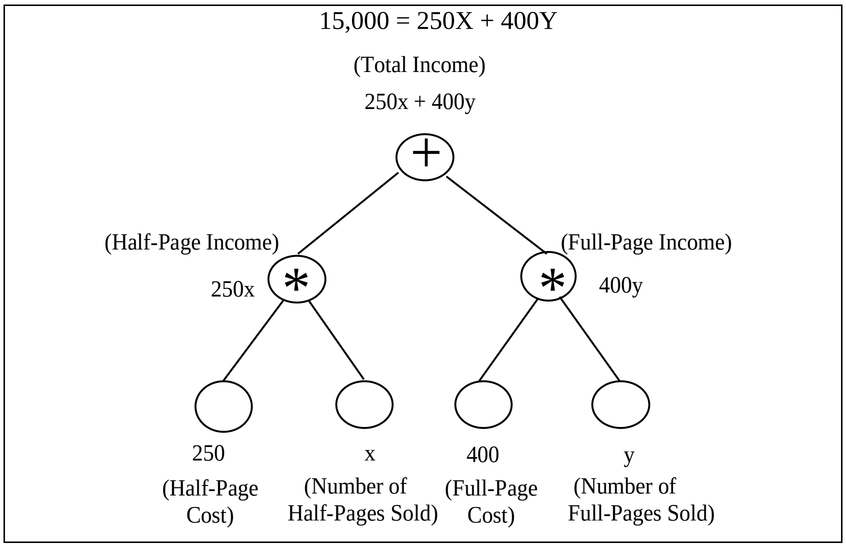

Fig. 2. Symbolic and hierarchical representation of the Fig. 1 problem situation.

The cognitive model for general linear form worksheet problems is tied to the underlying hierarchical structure of the problem situation. In solving each question, the model begins by identifying the given quantity. If the quantity has not yet been labeled, it finds and labels an open column, enters the unit, then enters the given value in the appropriate row. It then follows the hierarchical solution path through the question. For example, in question 2, the cognitive model recognizes it needs to compute the product *by* , then chains forward to the sum (constant) term, then to the *ax* product and finally finds the value of the variable quantity *x*. Students are *not* required to overtly follow these canonical solution paths. The tutor can recognize correct problem solving actions in any temporal order. However, if help is requested, the student is guided along a canonical path as described above.

3 Formative Evaluations

We completed two formative evaluations of the general linear form worksheet lesson. The first focused on students' understanding of the underlying hierarchical problem situations and the second focused on students' success in modeling the situations with a general linear form symbolic expression.

3.1 Understanding the Problem Situation

The first study examined both test data and tutor performance data to evaluate students' understanding of the underlying problem situation. In this study 19 students completed paper-and-pencil tests before and after completing the cognitive tutor lesson. These tests consisted of a problem situation and five numerical computation questions requiring students to traverse different paths through the underlying situation hierarchy. Students only wrote numerical answers. No worksheet was provided, and no algebraic representation of the situation was provided by, or for, the students. Achievement gains from pretest to posttest confirmed the effectiveness of the tutor lesson. Students averaged 44% correct on the pretest and 72% correct on the posttest. This 64 percent achievement gain is significant ($t = 3.08$, $p < .01$).

The order in which students filled in the worksheet cells in the tutor problems provides further evidence on students' understanding of the underlying hierarchical problem structure. The probability that the temporal order in which students fill the worksheet cells follows the underlying hierarchichal pathways rises from 0.46 for the first tutor problem to 0.76 for the final tutor problem. This increase is significant, $t = 2.81$, $p < .05$ and provides compelling evidence that students are learning the hierarchical relations in each problem situation.

3.2 Modeling the Problem Situation

A second study of the worksheet lesson examined how well students can generate symbolic models of problem situations and how well they understand these models. In this study 17 Cognitive Tutor Algebra II students completed two tests following completion of the general linear form worksheet lesson. One test presented a problem situation and five numerical questions, as in the earlier formative study, but also included a sixth question asking students to write a general equation that models the situation, e.g.,

> In this situation we need to earn $15,000 by selling half-page and full-page ads. Please write a general equation that represents this problem situation. Please use variables to represent the half-page ads and full-page ads and write an equation that shows how many of each are required to earn $15,000

The second test examined students' understanding of general linear form symbolic models. This test presented both a problem situation and a corresponding general linear form inequality and asked students to write natural language descriptions of the inequality components. An example test is presented in Fig. 3 with some appropriate answers displayed in italics.

Students in this study averaged 75% correct on the five numerical problem solving questions in test 1, replicating performance posttest performance levels from the first study. More importantly, students only scored 35% correct in writing a mathematical model of the situation, e.g., $250x + 400y = 15{,}000$. Accuracy rates for each of the five questions on the second test are displayed in Fig. 3. Students are very successful in describing the two constants, averaging 83% correct on questions (a) and (d). However, they are far less successful in describing any terms that include a variable, averaging 41% correct across questions (b), (c) and (e).

Suppose you have $10 to spend on refreshments at a movie theater. A box of popcorn costs $2.00 at the snack bar and a beverage costs $1.50. We can use the following inequality to represent the different snack combinations you can afford.

$$2x + 1.5\,y \leq 10$$

Please describe in your own words what the following five parts of the inequality represent in the snack bar situation.

(a) What does the number 1.5 represent in this situation? (85% correct)
The price of a beverage

(b) What does the variable y represent in this situation? (47% correct)
The number of beverages you purchase

(c) What does the term 2x represent in this situation? (41% correct)
The total you spend on popcorn

(d) What does the number 10 represent in this situation? (82% correct)
The amount of money you have to spend

(e) What does the sum 2x + 1.5y represent in this situation? (35% correct)
The total you spend on popcorn and beverages

Fig. 3. A symbolic model analysis paper-and-pencil test.

To examine the relationship between success in describing each of the five symbolic components on the second test and success in writing the full general linear form equation in the first test, we computed five chi-square analyses. Two of these tests were at least marginally reliable. Students who were able to describe the variable (*y*) were more likely to be able to write the equation ($\chi^2 = 10.43$, df = 3, $p < .05$) and students that were able to describe the product (*2x*) were marginally more likely to be able to write the equation ($\chi^2 = 6.80$, df = 3, $p < .10$).

Although this pattern of results does not prove a causal relationship, it is consistent with the hypothesis that helping students understand the mapping from the symbolic expression components to the problem situation will help students learn to write these symbolic models. This hypothesis inspired the design of a mathematical modeling tool, as described in the following section.

4 The Math Modeling Tool

We developed a Math Modeling Tool to scaffold students' understanding of the general linear form symbolic components and the mapping of these components to the problem situation. This tool is inspired by the analysis test described in the previous section. The tutor interface is displayed in Fig. 4. The tool presents a problem situation and a symbolic inequality that models the situation. The tool contains 8 rows for describing the mapping between situation components and symbolic model components. In this figure the analysis tool is seeded with each component in a hierarchical decomposition of the math model and the student fills in natural language descriptions of each component in the corresponding blank. Students select these natural language descriptions from a menu, as displayed in Fig. 5.

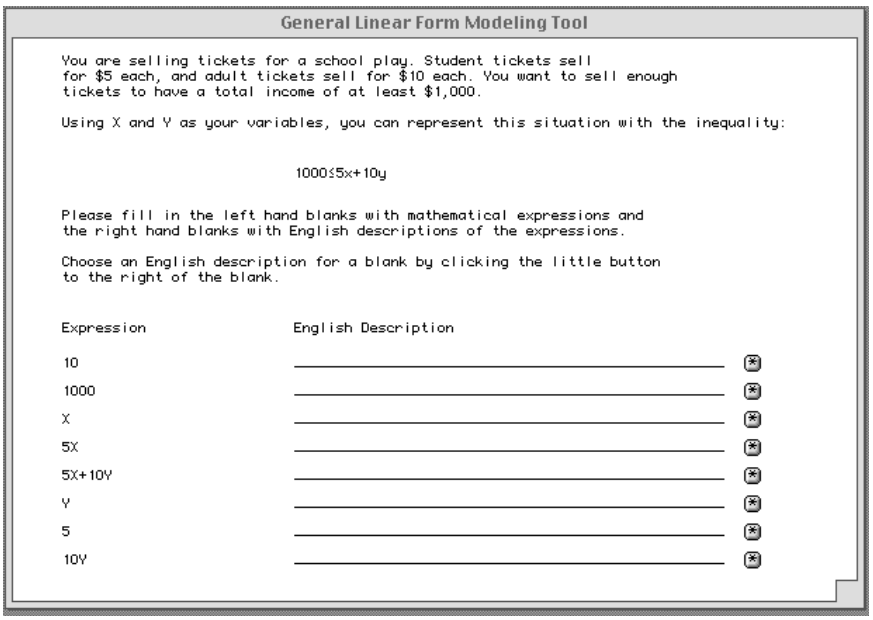

Fig. 4. The Math Modeling Analysis Tool Interface

Fig. 4 displays just one possible initial configuration of the tool. There are 17 slots in the tool that can be seeded or left blank at the beginning of a problem. The problem description is always presented, but the inequality can be seeded (as in Fig. 4) or left blank, each of 8 inequality components can be seeded (as in Fig. 4) or left blank and each of 8 component descriptions can be seeded or left blank (as in Fig. 4). If the inequality is left blank, students are required to fill it first. Students receive immediate accuracy feedback on each action they perform in the modeling tool and can request help in filling any slot. This tool was evaluated in the following study.

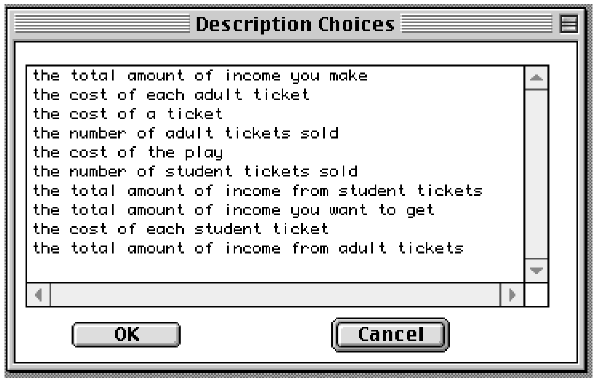

Fig. 5. The natural language description menu.

5 The Study

The Math Modeling Tool was piloted in a 4-day period near the end of the academic year. Eight-three Cognitive Tutor Algebra II students participated in the study. Fifty-six were enrolled in a Pittsburgh high school and 27 were enrolled in a nearby suburban high school.

5.1 Design

Lesson Design. Students completed 8 problems in the lesson. In the first and last problem all 17 slots in the math modeling tool were left blank. In these problems the student was required to type the inequality first, then fill in the remaining 16 blanks (8 inequality components and 8 natural language descriptions) in any order. In the remaining six problems the inequality was seeded. In two of these problems all 8 symbolic expression components were seeded and the student was required to provide the 8 descriptions. In two problems the 8 descriptions were provided and the student was required to type the corresponding expression components. In the other two problems a mix of expression components and descriptions were seeded and the student was required to fill in the missing blank in each pair.

Test Design. The same two tests were employed in this study as in the second formative evaluation described in Section 3.2. A problem solving test presented a problem situation description and students answered 5 numerical computation questions and generated a general linear form symbolic model. A model analysis test presented a problem description and general linear form inequality model and asked students to describe five hierarchical components of the model (as displayed in Fig. 3). Two versions of each test were constructed. Each version was completed as a pretest by one half of the students and a posttest by the other half of the students.

Procedure. Students completed the two pretests during an initial class period. The problem solving test was completed and removed before the analysis test was completed. Students worked through the 8 math modeling tutor problems over the next two days. Students completed the two posttests on the fourth day of the study, again completing the problem solving test before the analysis test.

5.2 Results

Table 1 displays the students' pretest and posttest performance in constructing and in describing components of general linear form symbolic models.

Mathematical Modeling and Numerical Computation (Test 1). Table 1 displays two measures of student accuracy in writing the general linear form symbolic model. The strict criterion requires students to code both sides of the inequality correctly *and* code the correct inequality relation between them, e.g., $1.5x+2y \leq 10$. The relaxed criterion requires students to code both sides of the relationship correctly, e.g., 1.5x+2y and 10, irrespective of the relational operator coded (a correct inequality, equation or incorrect inequality). Student accuracy in writing the general linear form model is reliably higher on the posttest than the pretest, both for the strict criterion, $z = 3.60$, $p < .001$ and the relaxed criterion, $z = 3.83$, $p < .001$).

Table 1. Student pretest and posttest performance (percent correct) in generating general linear form mathematical models and in analyzing model components.

	Test 1 Write a GLF Symbolic Model		Test 2 Write Natural language Description of Symbolic Model				
	Strict Scoring	Relaxed Scoring	Constants a	c	Variable Components x	ax	ax+by
	%corr	%corr	%corr	%corr	%corr	%corr	%corr
Pretest	7%	40%	86%	47%	64%	54%	45%
Posttest	30%	68%	94%	57%	71%	49%	68%

The mathematical modeling tool did not affect student performance in completing the five numerical computation problems. Students averaged 64% correct on the pretest and 68% correct on the posttest and this difference is not reliable.

Natural Language Descriptions (Test 2). Table 1 also displays the impact of the Math Modeling Tool on students' descriptions of the symbolic expression components. The overall increase in correct descriptions from 59% on the pretest to 68% on the posttest is reliable, $t(82) = 2.81$, $p < .01$. The improvement in describing the sum expression (e.g., 1.5x+2y) is reliable, $z = 3.34$, $p < .001$. The pretest to posttest performance change for the other four individual questions is not significant.

6 Discussion

The Math Modeling Tool significantly improved student success in writing general linear form mathematical models and significantly improved achievement in describing symbolic model components, but did not affect achievement in solving numerical computation problems. This pattern is consistent with our preliminary interpretation of the formative evaluations described in section 3. Students' success in writing symbolic models in the earlier studies was primarily limited by their understanding of the symbolic representations and their mapping to the problem situations, rather than their comprehension of the underlying problem situation. This study helps demonstrate the importance of ongoing formative evaluations and iterative refinements of intelligent tutoring environments.

Note that after using the Math Modeling Tool, students are more successful in generating the two sides of the symbolic inequalities than at encoding the inequality sign relating them. While the problem statements display the inequality sign in the symbolic models, students are not required to actively process this component of the models. Hence this study demonstrates an important design principle: declarative presentation of information is not sufficient to support learning. Students need to actively engage with knowledge structures to learn how to use them. However, this study suggests that actually performing a skill (e.g., writing math models), is not the only activity that supports learning the skill. Other problem solving activities that support understanding of a task can help scaffold performance of the task.

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