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ABSTRACT

This paper presents a developmental model of students' acquisition of competence in quantitative and algebraic problem solving. A key notion underlying the developmental model is a distinction between grounded and abstract representations. Grounded representations, like story problems, are more concrete and familiar, closer to physical objects and everyday events. Abstract representations, like symbolic equations, are concise and easy to manipulate, but are distanced from any physical objects of reference. The complementary computational characteristics of grounded and abstract representations lead to hypotheses about the order of skill acquisition. In prior research, the authors demonstrated that early in the development of algebraic competence, the advantages of grounded representations outweigh those of abstract representations--for simpler problems, students are better at story problems than the analogous equations. This paper presents two studies that test the hypothesis that later in algebra development, the advantages of abstract representations emerge--for more complex problems, students are better at equations than the analogous story problems. Includes 6 tables, 7 figures, and 16 references. (Author/WRM)

A Developmental Model of Algebra Problem Solving:

Trade-offs Between Grounded and Abstract Representations

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ABSTRACT

We present a developmental model of students' acquisition of competence in quantitative and algebraic problem solving. A key notion underlying our developmental model is a distinction between grounded and abstract representations. Grounded representations, like story problems, are more concrete and familiar, closer to physical objects and everyday events. Abstract representations, like symbolic equations, are concise and easy to manipulate, but are distanced from any physical objects of reference. The complementary computational characteristics of grounded and abstract representations lead to hypotheses about the order of skill acquisition. In prior research (Koedinger & Nathan, 1999), we demonstrated that early in the development of algebraic competence, the advantages of grounded representations outweigh those of abstract representations – for simpler problems, students are better at story problems than the analogous equations. This paper presents two studies that test the hypothesis that later in algebra development, the advantages of abstract representations emerge – for more complex problems, students are better at equations than the analogous story problems.

INTRODUCTION

This paper presents a developmental model of students at different levels of competence in quantitative and algebraic problem solving. In prior research, we employed the difficulty factors assessment (DFA) methodology to explore early algebra problem solving, and identified effects that contradict common beliefs and current instructional practices (Koedinger & Tabachneck, 1995; Koedinger & MacLaren, 1997; Koedinger & Nathan, 1999). In this paper, we review these results and present new results about student problem solving at higher levels of competence. Together these results on difficulty factors in algebra problem solving provide a picture of student development from arithmetic competence through various distinguishable levels of algebraic competence.

A key notion underlying our developmental model is a distinction between grounded and abstract representations. Grounded representations are ones that are more concrete, closer to physical objects and everyday events. In the context of quantitative reasoning, real world problem situations or "story problems" are more grounded than symbolic equations because they use familiar words and refer to familiar physical objects and events[1]. For example, consider the following story problem.

> Ted works as a waiter. He worked 6 hours in one day and also got $66 in tips. If he made $81.90 that day, how much per hour does Ted make?

Given some experience with money and waiters, the words, objects and events described in this problem are relatively familiar to students. Students' understanding of the quantitative relationships described is thus grounded in these familiar terms.

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1 How grounded a particular problem is for a particular student depends on that student's past experiences with the particular objects and events in a problem as well as the words or symbols used to refer to them. To the

Abstract representations, in contrast, are short and concise, leaving out any direct indication of the physical objects and events being referred to. Consider the following algebraic representation of the story problem above.

$$x * 6 + 66 = 81.90$$

The equation is clearly shorter and more concise than the story problem above. Besides the numbers common to both problems (6, 66, 81.90) there are only four other characters in the equation (x, *, +, =) whereas there are 100 other characters in the story. What is left out, however, is any reference to the familiar objects and events like hourly wages and tips. Furthermore, the terminology is different. The "words" in the algebraic sentence (i.e., x, *, +, =) are less familiar than the phrases expressing analogous meanings in the story (i.e., how much, hours in one day, also got, he made).

The difference between grounded and abstract representations is not as discrete as these examples might indicate. There are multiple levels of intermediate groundedness or abstractness. Consider the following word problem, which is devoid of situational content:

> Starting with some number, if I multiply it by 6 and then add 66, I get 81.90. What number did I start with?

It is intermediate in abstractness between the story and the equation, having 78 characters besides the numbers. It is also intermediate in its groundedness. While it is missing familiar references to money and waiters, it contains words that are more familiar (i.e., some number, multiply, add, get) than the characters expressing analogous meaning in the equation (i.e., x, *, +, =).

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extent that there is substantial overlap in common experience across students we can expect general effects of grounding. Our results suggest that this assumption holds.

Trade-offs in Representational Advantages

Grounded and abstract representations have different strengths and weaknesses for reasoning and computation (see Table 1). Grounded representations are familiar and thus, unlike abstract representations, do not put demands on long term memory to learn and remember new symbols, syntax and semantics. For example, in the word problem above, the words (e.g., some number, gets) and the grammar (e.g., use of "then") are familiar. However, to correctly comprehend the equations, students must learn new "words" (e.g., x, =) and grammar (e.g., order of operations). Grounded representations tend to be error-resilient, in the sense that students are less likely to make errors, and more likely to detect them when they are made. With abstract representations, errors often go unnoticed. For example, for the "waiter" problem presented above, students seldom add the number of hours worked to the amount of tips. However, for the corresponding equation, students frequently add 6 + 66.

Insert Table 1 about here

Although they are error-prone, abstract representations also have several advantages. Working with an abstract representation can be fast and efficient. Further, abstract representations put fewer demands on working memory than grounded representations, for a number of reasons. First, working with abstract representations is less resource-intensive than working with grounded representations, since one need not keep track of the referents of all the symbols while solving the problem. Second, it is easier to make use of paper as an external memory aid because the conciseness of an abstract representation means many fewer characters need to be written down, for example, in the process of equation solving. Third, it is easier to visually process and mentally imagine manipulations of quantitative constraints (cf. Kirshner, 1989), for example, the combination of like terms.

Representational Advantages in Algebra Development

Koedinger and Nathan (1999) have argued that, as students make the transition from arithmetic to algebra (in the early high school years), the advantages of grounded representations outweigh their disadvantages. They asked high school students to solve problems that were presented in either a story format, a word-equation format, or a symbolic format (see Table 2)[2]. As shown in Figure 1, they found that students succeeded more often both on story problems and on word equations than on the corresponding symbolic equations. Thus, students performed better when problems were presented in a grounded, verbal format than when problems were presented in symbolic form. This finding suggests that the advantages of grounded representations are especially important in the early stages of the development of algebraic skill.

Koedinger and Nathan (1999) presented evidence for two converging explanations for this pattern of results. First, students were less successful on symbolic equations than one might have expected. Students' errors in the symbolic format often revealed serious difficulties with the syntax and semantics of algebra. As seen in Figure 2a, students made errors both in comprehending and in manipulating algebraic expressions.

Second, students were more successful on story problems than one might have expected. In contrast with normative expectations, students often did not solve the story and word problems by converting them to equations and then solving the equations (see Figure

[2] Table 2 and Figure 1 also illustrate a second dimension: whether the problem unknown is a *result* of the arithmetic operations described (result-unknown) or at the *start* of the arithmetic operations (start-unknown). See Koedinger & Nathan (1999) for more details on this distinction and its impact on student performance.

2b). Instead, students more often used informal strategies, not involving algebraic symbols, to "bootstrap" their way to correct answers for the story problems. Students sometimes used an iterative guess-and-test procedure to arrive at a solution (see Figure 2c), and sometimes worked backward through the constraints provided in the problem, "unwinding" the constraints (see Figure 2d) rather than using algebraic manipulation.

Based on these findings, Koedinger and Nathan proposed a developmental model of early algebra development in which competence in grounded representations emerges before competence in abstract representations. As seen in the top half of Figure 3 (levels 0-3), students tend to succeed on problems presented in verbal format before they succeed on problems presented in equation format. Interestingly, some students exhibit competence in solving early algebra problems (start-unknowns) in verbal form prior to success on arithmetic problems (result-unknowns) in equation form (see level is 2b in Figure 3).

These previous findings suggest a clear superiority for grounded representations over abstract ones. However, if this is the case, then why learn symbolic algebra at all? We suggest that the advantages of grounded representations hold true at the early stages of algebra development, because students are working with simple problems. However, later in development, as students begin to solve more complex problems, we expect the advantages of the symbolic format to emerge.

As Koedinger and Nathan (1999) argued, early in the development of algebra skill, students succeed within grounded representations by using informal strategies, and they fail within abstract representations because they fail to understand the foreign language of algebra. We hypothesize that, later in development, this pattern may reverse. As students gain skills for equation solving, their success at symbolic problems improves. At the same time, as problems become more complex, the limitations of informal strategies may begin to emerge. The guess-and-test strategy (described above, see Figure 2c) becomes time-consuming and error prone with complexity, for instance, when answers are multi-digit decimals. The unwind strategy (see Figure 2d) depends on being able to invert operators and work backwards toward the problem unknown. However, the unwind strategy is thwarted when the problem unknown is referenced more than once. For instance, consider the following "multiple-unknown" problem (shown in both story and equation representations):

> Roseanne just paid $38.24 for new jeans. She got them at a 15% discount. What was the original price?
>
> $X - 0.15X = 38.24$

A student cannot perform the unwind strategy[3] since the "last" operation performed, the subtraction of the discount from the original price, involves two unknown terms (i.e., x and 0.15x). Since neither of these is constant, the student cannot unwind, at least not correctly (students will try, for instance, by adding 15% of 38.24 to 38.24).

Withcut an effective informal strategy for solving a complex problem, students can turn to the normative algebraic strategy of translating to an equation and solving that equation (cf. Mayer, 1985). In this strategy, equation solving is a sub-problem of story problem solving, and thus story problems will necessarily be harder to the extent that students have difficulties in translating stories to equations. This translation process is well known to be notoriously difficult for students (e.g., Heffernan & Koedinger, 1997, 1998; Mayer, 1982; Nathan, Kintsch, Young, 1992).

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[3] Although the unwind strategy does not work on multiple-unknown problems, other informal strategies can be successful on such problems. For instance, by realizing that the sale price is 85% of the original the student can, in effect, mentally combine the like terms X and 0.15X. This informal constraint combination is more sophisticated and memory demanding than the simpler unwind strategy.

Taken together, these ideas suggest that, at later stages in the development of algebraic skill and for more complex problems, students may succeed with more abstract representations but fail with more grounded ones. Hence, a "symbolic advantage" will emerge as learners develop competencies for solving more complex problems. In this paper, we present two experiments that test this "symbolic advantage" hypothesis in two samples of college students. Experiment 1 investigates whether there is a symbolic advantage for students with fairly weak mathematics skills, that is, college students in an algebra review course. Experiment 2 investigates whether there is a symbolic advantage for more highly skilled students.

EXPERIMENT 1

Method

Participants

The study participants were 153 developmental math students at a state university in southern California in an algebra course. Of these students, 43 were in a more basic "Elementary Algebra" course (content similar to a high school algebra 1 course) and 110 were in a more advanced "Intermediate Algebra" course (content combines high school algebra 1 and 2 topics).

Insert Table 4 about here

Procedure

All students were given 20 minutes within class time to complete a six-item difficulty factors assessment. Each student was randomly assigned one of six alternative forms. The forms were designed to contrast the two difficulty factor dimensions, representation and problem complexity, illustrated in Table 3. Following the Koedinger & Nathan (1999) study described above, we manipulated problem representation, creating mathematically equivalent problems in both story and equation representations. Two types of story problems were used: 1) more natural "story-implicit" problems in which arithmetic operations are expressed through everyday verbs (e.g., "kept") and 2) more equation-like "story-explicit" problems in with the arithmetic operations are explicitly expressed (e.g., "subtracted"). Examples of these story problems are illustrated in the first two columns of Table 3. The third column illustrates the equation representation.

The rows in Table 3 illustrate the problem complexity dimension. The simpler start-unknown problems, illustrated in the first row, are just like the start-unknowns used in the Koedinger and Nathan (1999) – we used three of the four start-unknown problems used there. The more complex problems are multiple-unknowns.

Insert Table 4 about here.

For each level, there were three different cover stories (illustrated in Table 4). The three start-unknown cover stories: donut, waiter, and lottery are shown at the bottom of Table 4. The three multiple-unknown cover stories: students, exchange, and discount are shown at the bottom of Table 4. Combining these six cover stories with the three representation types yielded 18 different problems. These problems were distributed on the six forms such that: 1) each problem appeared on two different forms with its position on those forms counterbalanced (i.e., if it was in the first position on one form it was sixth on the other) and 2) all six cover stories appeared on each from with each of the three representations appearing at both levels of complexity.

Results

Complexity Leads to an Advantage for Symbols

Figure 4 shows the average correct on the six problem categories illustrated in Table 3. On the simpler start-unknown problems, students performed better with the more grounded story representations than with the more abstract equation representations. In contrast, on the more complex multiple-unknown problems, students performed better with the abstract equation representation than with the story representation. A two factor within-subjects ANOVA indicates a statistically significant main effect of problem complexity, $F(1, 152) = 203$, $p < .0001$, and an interaction between problem complexity and representation, $F(2, 304) = 30$, $p < .0001$. An item analysis shows that this interaction effect also generalizes across items, $F(2, 8) = 4.59$, $p < .05$.

That story start-unknowns are easier than equation start-unknowns for these college students is consistent with the high school results presented above. However, there are some important differences. In general, the college students' performance is much better than the high school students' (69% vs. 49%, respectively, on these problems). Looking at the three start-unknown cover stories we used and their corresponding equations (see the first three rows of Table 4), we see that for the most part these student have mastered two-operator start-unknown problems. However, whereas their story problem solving competence is uniformly high (averaging 79%, 70%, and 81% on the three problem types), their equation solving competence is spotty. For the two problems in the familiar "mx + b = y" form, students perform just as well on the equations (77% and 79%). However, for the equation in less familiar "(x – c) / n = y" form, students perform much worse on the equation (23%).

The results for the multiple-unknown problems are quite different. For all three cover stories and their corresponding equations (see the last three rows of Table 4). students performed better on the equations than on the story problems.

Illustrating Developmental Levels

College student performance on the simpler start-unknown problems is consistent with the high school student results. As illustrated in levels 2b and 3 of our developmental model (Figure 3), students are more likely to be successful first on start-unknowns represented verbally than start-unknowns represented as equations. Some of our college students are at these early stages of algebra development. Two such students are illustrated in Figure 5 – they are both at level 2b. These examples illustrate that performance is driven by the problem representation, and not by other problem features such as the cover story or the underlying arithmetic. One student succeeds on the story version of a start-unknown problem (donuts for student 87, lottery for student 46) while the other student fails on the equation version of that same problem. Note that both students succeed on the story problem without using an equation. Instead, each successfully applies the informal unwind strategy. However, on the equation both students attempt formal algebra manipulation, but fail.

Other students in our sample showed more advanced algebraic competence corresponding with developmental levels 4 and 5 in Figure 3. The students shown in Figure 6 are at level 4 and illustrate the symbolic advantage. These students are successful on multiple-unknown equations, but fail on multiple-unknown story problems. As above, these two students illustrate the impact of the problem representation. One student succeeds on the equation version of a multiple-unknown problem (discount for student 19, exchange for student 82) while the other student fails on the story version of the same problem. Note that both students have difficulty in translating the story to an equation, but when given an equation have now developed the competence to successfully take advantage of this symbolic representation.

EXPERIMENT 2

Experiment 2 examined whether the "symbolic advantage" observed among developmental math students in Experiment 1 also held in a more mathematically sophisticated sample.

Method

Participants

Sixty-five undergraduate students participated in partial fulfillment of a course requirement. Students' self-reported Math-SAT scores ranged from 510 to 800, with a mean score of 719.

Procedure

All students completed a difficulty-factors assessment designed to address a variety of algebra skills including problem solving. Twenty of the students completed the assessment in individual sessions, and provided talk-aloud protocols while doing so. The remaining forty-five students completed the assessment in a written test that was administered in a group setting.

The assessment was designed to evaluate four types of algebra competencies: (1) solving problems presented in an equation format, (2) solving problems presented in a story format, (3) drawing generalizations from a table of (x,y) pairs, and (4) translating expressions between different representational formats (e.g., generating a story problem for a symbolic expression, or generating an equation for a story problem). The analyses presented here focus on students' competencies in problem solving; students' competencies in translation and generalization will not be considered.

All of the problem-solving item types were presented in both story and equation format. All of the story problems used in Experiment 2 were of the more natural, implicit operator type. As seen in Table 5, the assessment included both problems with a single unknown (two types) and problems with multiple unknowns (three types).

Results

Complexity Leads to an Advantage for Symbols

As in Experiment 1, we hypothesized that, for the simpler, single-unknown problems, students would perform better with the more grounded story representations than with the more abstract symbolic representations. In contrast, for the more complex multiple-unknown problems, we expected students to perform better with the abstract, symbolic representations. As seen in Figure 7, both of these predictions were borne out, yielding a statistically significant interaction between problem complexity (single vs. multiple unknowns) and representational format (equation vs. story), $F(1, 64) = 14.32$, $p < .001$. Planned contrasts indicated that students showed a statistically significant story advantage for the single-unknown problems ($F(1, 64) = 6.11$, $p < .02$), as well as a statistically significant symbolic advantage for the multiple-unknown problems ($F(1, 64) = 13.39$, $p < .001$). These findings replicate those of Experiment 1. As expected, there was also a substantial effect for problem complexity, $F(1, 64) = 15.96$, $p < .001$. Further, the same pattern held both for students who completed the assessment in protocol form, and for students who completed the assessment in written form. That is, there was no three-way interaction with assessment procedure (protocol or written), $F(1, 64) < 1$.

Results for individual problems are presented in Table 6. As expected, for both types of single-unknown problems, students showed a story advantage, and for all three types of multiple-unknown problems, students showed a symbolic advantage. These results support the claim that, as problem complexity increases, the advantages of the symbolic representation begin to outweigh the advantages of the more grounded representation.

Participants used formal algebraic strategies more often for multiple-unknown problems than for single-unknown problems, for both story and equation representations. When students used informal strategies for single-unknown problems, they were very likely to be accurate (90% overall; 88% for equations and 93% for stories). In contrast, when students used informal strategies for multiple-unknown problems, they were much less likely to be accurate, especially on story problems (59% overall; 89% for equations and 50% for stories). To test this pattern statistically, we limited the analysis to students who used informal strategies for some problems in each category (single- vs. multiple-unknown). As expected, when students used informal strategies, they were more successful applying them to single-unknown problems (92%) than to multiple-unknown problems (62%), $t(19) = 2.87$, $p < .01$.

DISCUSSION

These finding extend previous work that showed that, early in the development of algebra skill, students often succeed with grounded representations, but fail with more abstract ones (Koedinger & Nathan, 1999). The present experiments have shown that, on more complex problems, students demonstrate an advantage for symbolic representations. Thus, early in development, and on simple problems, students show a "verbal advantage." On more complex problems, they show a "symbolic advantage."

Why does the symbolic advantage emerge for complex problems? Our data indicate that students succeed on complex equations because they have mastered equation solving, at least in part. However, students fail on complex story problems, because such problems thwart some of their readily-available informal strategies (e.g., unwind). Instead of applying informal strategies, students attempt to translate complex story problems into equations, and they often fail on this translation process, leading to incorrect problem solutions.

Trade-offs in Representational Advantages

Our data support the idea that grounded and abstract representations have different strengths and weaknesses (see Table 1). For simple problems, the strengths of grounded representations outweigh their weaknesses. Grounded representations help students to avoid and/or detect errors (e.g., adding hours to tips). Grounded representations also lead to high demands on working memory, but, in the case of simple problems, overall working-memory demands are still fairly low. For simple problems, the disadvantages of working-memory demands are not enough to outweigh the advantages in reliability, even for the sophisticated students in Experiment 2. Hence, students show an advantage for the grounded representations.

For more complex problems, the disadvantages of grounded representations outweigh their strengths, and conversely, the strengths of abstract representations outweigh their weaknesses. Complex problems prove too difficult to manipulate in grounded form, because of the high working-memory load involved with grounded representations. Although the more concise, symbolic representations are error-prone, the advantages of easy manipulation outweigh this disadvantage. Thus, for complex problems, students show an advantage for symbolic representations.

Thus, our data indicate that there are trade-offs between different representational formats. We have found additional evidence for such trade-offs in prior research with college students solving story problems (Koedinger & Tabachneck, 1995). Students are especially likely to solve a difficult problem successfully if they use multiple strategies, including one that capitalizes on the strengths of a grounded representation (e.g., unwind), and one that capitalizes on the strengths of an abstract representation (e.g., translate to an equation and solve using algebraic manipulation).

INSTRUCTIONAL IMPLICATIONS

Understanding these trade-offs between grounded and abstract representations can help guide the design of effective instruction. This work investigated grounded and abstract representations as part of difficulty-factors assessments with the goal of identifying factors that affect student performance at problem solving. The two DFA experiments converge with others reported in past work (Koedinger & Nathan, 1999) to highlight the importance of representational format (e.g., story vs. equation) and problem complexity (e.g., single vs. multiple unknowns) as important factors that influence students' performance.

Research with the DFA methodology can be used to construct models that specify the sequence in which skills are learned, as shown in Figure 3. Early in the course of learning algebra, students tend to acquire skills for solving simple verbal problems before they acquire skills for solving comparable symbolic problems. Later in the course of learning algebra, students tend to acquire skills for solving complex symbolic problems before they acquire skills for solving comparable verbal problems.

Our analysis suggests that one reason for this reversal in order of acquisition has to do with the competencies required for success. For simple problems, students succeed by using informal strategies that work well within grounded representations. For more complex problems, students cannot use informal strategies to succeed within the grounded representation. Instead, students must translate to a formal, symbolic representation and solve using algebraic manipulation. Thus, different competencies are required to succeed within the grounded representation for complex problems than for simpler problems.

Note that the levels in our developmental model indicate strong tendencies, but not a strict ordering. Students sometimes display competencies out of order; for example, students can sometimes display greater success on symbolic start-unknown problems than on structurally equivalent verbal problems – in other words, appearing to perform at level 3 (see Figure 3) without having level 2b competence. We consider such out-of-order competencies to be "ungrounded," in the sense that they have been learned of themselves and are not grounded in prior knowledge. Ungrounded competencies are likely to be brittle, in the sense that are unlikely to be retained well, and they may not be applied in novel problem situations.

Overcoming "expert blindspot" in instructional design

We believe that developmental models like the one we have proposed can be useful in designing effective instruction. Such models can provide guidance about the appropriate sequencing of topics in a curriculum, and can help curriculum designers to overcome the "expert blindspot" that may occur when experts' ideas about what may be difficult for students differ from the reality. Our model indicates that, at least for the early parts of learning algebra, the sequence of topics in most current algebra textbooks may be inaccurate. Instead of learning to solve equations first and story problems later, our analysis suggests that it might be more effective for students to learn to solve story problems first, and then bridge from this understanding to learn to solve equations.

Bridging instruction

We suggest that instruction that bridges from students' informal or grounded knowledge may be more effective than instruction that focuses directly on abstract representations. Several recent experiments support this idea. For example, Koedinger and Anderson (1998) compared two versions of a computer-based Cognitive Tutor (Anderson, Corbett, Koedinger & Pelletier, 1995) that contrasted a bridging approach with a textbook instructional approach to algebraic symbolization. In the bridging version, the cognitive tutor encouraged students to solve story problems numerically (using grounded strategies) prior to attempting to translate the same problem to algebraic symbols. The Cognitive Tutor supports students forming algebraic sentences (e.g., "42 * h + 35") by helping them to induce or generalize from examples of analogous arithmetic procedures (e.g., "42 * 3 + 35" and "42 * 4.5 + 35") they have just performed. The experimental comparison demonstrated that students learned more from this bridging approach than from a textbook approach that had students symbolize prior to problem solving. In other words, bridging from grounded problem-solving performance helped students to better learn to translate to the abstract symbolic representation.

In another study, Alibali (1999) provided fourth-grade students with a lesson that drew links between an equation such as 3 + 4 + 5 = __ + 5 and students' grounded, action-based knowledge about a teeter-totter. The bridging lesson led to successful learning and broad transfer for a subset of students – broader transfer than observed among students who learned a procedure for solving the problems correctly.

As another example, Nathan (1998; Nathan, Kintsch, & Young, 1992) designed a computer-based learning environment, called "ANIMATE", in which students attempted to symbolize word problems, and their symbolization attempts "drove" an animator that visually displayed the outcome of their attempt. Experience with ANIMATE led to greater success at symbolization than traditional instruction on symbolizing.

In general, experimental research to date supports the claim that instruction that bridges from students' existing knowledge is effective. We are currently exploring the effects of bridging instruction in a series of ongoing classroom-based experiments in a middle school mathematics class. One approach we are taking is asking students to solve problems and then to "summarize" their solution procedures in symbolic form (Nathan & Koedinger, in press). This process is designed to help students ground symbolic sentences, or give them meaning, by connecting them to students' prior knowledge of analogous actions and verbal forms.

We suggest that to design bridging instruction well it is crucial to understand students' competencies and the order in which they are acquired. Toward this end, our work has shown that early in the development of algebraic skill, students acquire skills for solving simple verbal problems before comparable symbolic problems, whereas later in the course of learning algebra, students acquire skills for solving complex symbolic problems before comparable verbal problems. To understand how to bridge from students' existing knowledge, it will be essential to understand the trade-offs between different types of representations and how these trade-offs come into play over the course of development.

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Table 1. Trade-offs in computational characteristics of more grounded versus more abstract representations.

	Demands on long-term memory	Reliability	Demands on working memory	Efficiency	**Examples**
Grounded	***Already known***	***Error resilient***	**Wordy & "heavy"**	**Slow**	Stories
					Words
Abstract	**Hard to Learn**	**Error prone**	***Concise & "light"***	***Fast***	Equations

Table 2. Six Problem Categories Illustrating Difficulty Factors Early in Algebra Development.

STORY PROBLEM	WORD EQUATION	SYMBOLIC EQUATION
	RESULT-UNKNOWN	
When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much does Ted make per hour?	Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?	Solve for x: (81.90 - 66)/6 = x
	START-UNKNOWN	
When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he had earned $81.90. How much does Ted make per hour?	Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?	Solve for x: x * 6 + 66 = 81.90

Table 3. Six Problem Categories Illustrating Two Difficulty Factors Later in Algebra Development: Representation (Story-implicit, Story-explicit, or Equation) and Problem Complexity (Start-unknowns, Multiple-unknowns)

STORY-IMPLICIT	STORY-EXPLICIT	EQUATION
START-UNKNOWNS (a kind of SINGLE-UNKNOWN)		
Mom won some money in a lottery. She kept $64 for herself and gave each of her 3 sons an equal portion of the rest of it. If each son got $20.50, how much did Mom win?	After hearing that Mom won a lottery prize, Bill took the amount she won and subtracted the $64 that Mom kept for herself. Then he divided the remaining money among her 3 sons giving each $20.50. How much did Mom win?	Solve for the unknown value, X: $(X - 64) \div 3 = 20.50$
MULTIPLE-UNKNOWNS		
Roseanne just paid $38.24 for new jeans. She got them at a 15% discount. What was the original price?	Roseanne bought some jeans on sale for $38.24. To figure that sales price, the salesperson took the original price, multiplied it by the 15% discount rate, and then subtracted the outcome from the original price. What was the original price?	Solve for the unknown value, X: $X - 0.15X = 38.24$

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Table 4. Problems and results for Experiment 1. The Story problem shown is the Implicit Operator version, but the data is the mean proportion correct for both versions.

Stories (Implicit & Explicit)		**Equations**	
Laura bought 7 donuts and paid $0.12 extra for the box to hold them. If she paid $2.57 total, what is the price per donut?	.79	7X + .12 = 2.57	.77
Ted works as a waiter. He worked 6 hours in one day and also got $46 in tips. If he made $65.50 that day, how much per hour does Ted make?	.70	6X + 46 = 65.50	.79
Mom won some money in a lottery. She kept $64 for herself and gave each of her 3 sons an equal portion of the rest of it. If each son got $20.50, how much did Mom win?	.81	(X - 64) ÷ 3 = 20.50	.23
There are 38 students in class. If there are 6 more girls than boys, how many boys are in the class?	.52	X + (X + 6) = 38	.69
You are in Paris, France and you want to exchange your dollars for French Francs (FF). The first exchange store gives you 5.7 FF per dollar, but charges 22 FF for each exchange. The second exchange store gives you 5.4 FF per dollar, and does not charge a fee. When are the charges from the two stores the same, in other words, what amount of dollars results in the same charge from both stores?	.06	5.7X - 22 = 5.4X	.44
Roseanne just paid $38.24 for new jeans. She got them at a 15% discount. What was the original price?	.05	X - 0.15X = 38.24	.31

Table 5. Problems used in Experiment 2.

Problem Type	Story Problem	Equation
	Single Unknown	
Result Unknown	Anne is in a rowboat on a lake. She is 800 yards from the shore. She rows towards the shore at a speed of 30 yards per minute. How far is Anne from the shore after 23 minutes?	$800 - 30 * 23 = y$
Start Unknown	Kim is saving up for a mountain bike that costs $600. She earns $20 per week by babysitting every Saturday afternoon. She can save all of the money that she earns. If Kim only needs to save $260 more, how many weeks has she already saved?	$600 - 20 * x = 260$
	Multiple Unknown	
Same Side (easy)	There are 38 students in class. If there are 6 more girls than boys, how many boys are in the class?	$X + (X + 6) = 38$
Same Side (hard)	Roseanne just paid $38.24 for new jeans. She got them at a 15% discount. What was the original price?	$X - 0.15X = 38.24$
Different Sides	You are in Paris, France and you want to exchange your dollars for French Francs (FF). The first exchange store gives you 5.7 FF per dollar, but charges 22 FF for each exchange. The second exchange store gives you 5.4 FF per dollar, and does not charge a fee. When are the charges from the two stores the same, in other words, what amount of dollars results in the same charge from both stores?	$5.7X - 22 = 5.4X$

Table 6. Proportion of Participants who Solved each Problem Correctly in Experiment 2

Problem Type	Representational Format	
	Story Problem	Equation
	Single-Unknown Problems	
Result Unknown	0.97	0.89
Start Unknown	0.94	0.88
	Multiple-Unknown Problems	
Same Side (easy)	0.86	0.94
Same Side (hard)	0.63	0.74
Different Sides	0.82	0.94

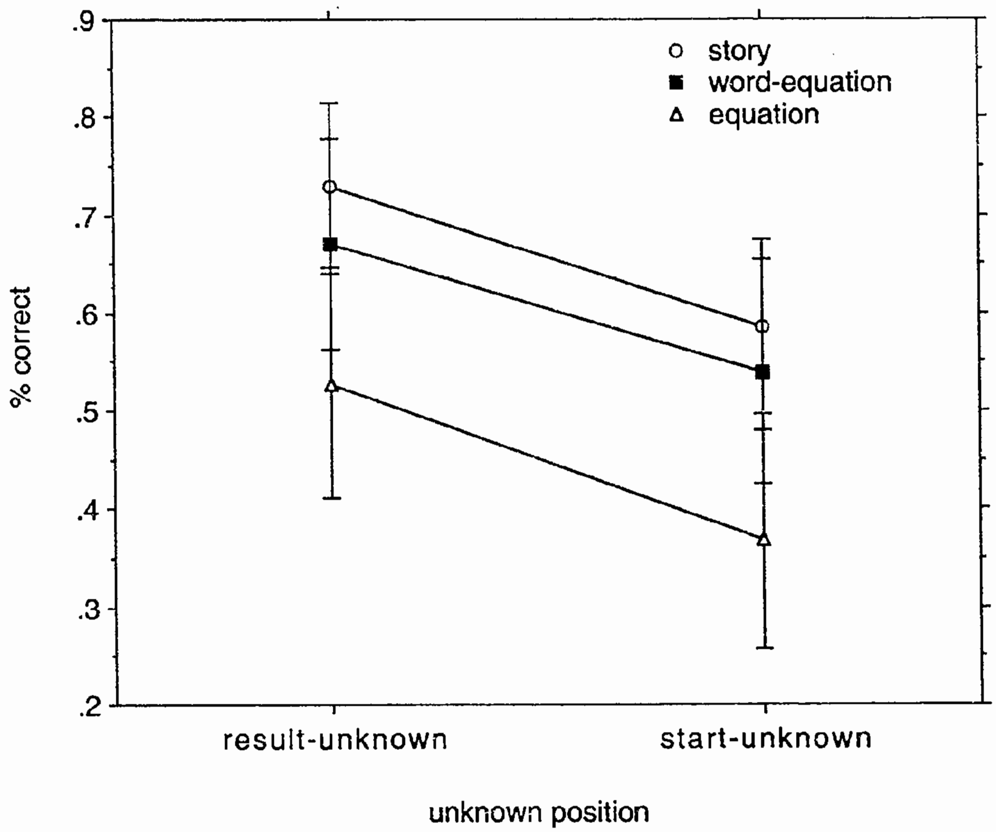

Figure 1. Beginning algebra students find simple two-operator story and word problems *easier* than the equivalent equations.

2. Solve for x:

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x * .37 + .22 = 2.81
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.37
.22
.59

2.22
+.59
2.81

$\boxed{2.81}$

1. Solve for x:

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20 * 3 + 40 = x
```

-40 -40

-20 x 3 = x

+20 +20

3 = 20x

20 20

2. Solve for x:

```
x * 25 + 10 = 110
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-10 -10

x X 15 = 110

-15 -15

x = 95

a. Difficulties with the syntax and semantics of symbolic algebra: Order of operations and identifying the sides of an equation.

8. After buying donuts at Wholey Donuts, Laura multiplies the number of donuts she bought by their price of $0.37 per donut. Then she adds the $0.22 charge for the box they came in and gets $2.81. How many donuts did she buy?

.37X + .22 = 2.81

-.22 -.22

$\frac{.37X}{.37} = \frac{2.59}{.37}$

.37)2.59 = 7

X = 7

b. The normative strategy: Translate to algebra and solve algebraically.

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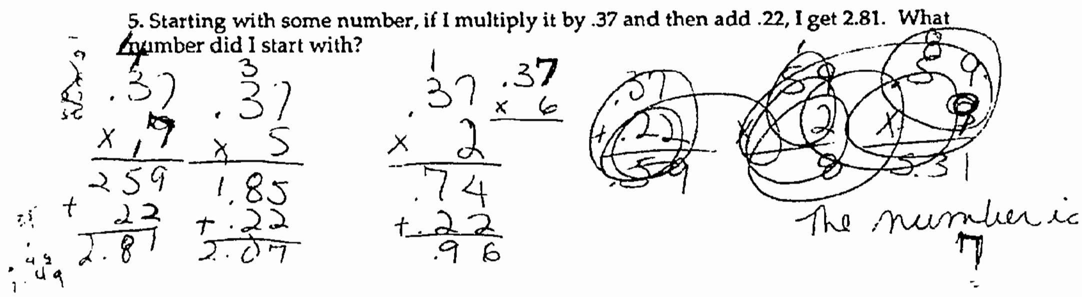
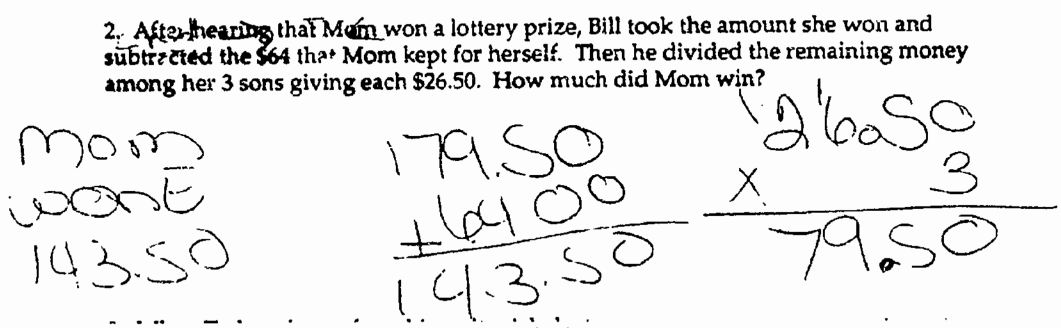

5. Starting with some number, if I multiply it by .37 and then add .22, I get 2.81. What number did I start with?

c. The guess-and-test strategy.


2. After hearing that Mom won a lottery prize, Bill took the amount she won and subtracted the $64 that Mom kept for herself. Then he divided the remaining money among her 3 sons giving each $26.50. How much did Mom win?

d. The unwind strategy.

Figure 2. Examples of strategies and errors from high school algebra students solving start-unknown story problems, word equations, and equations.

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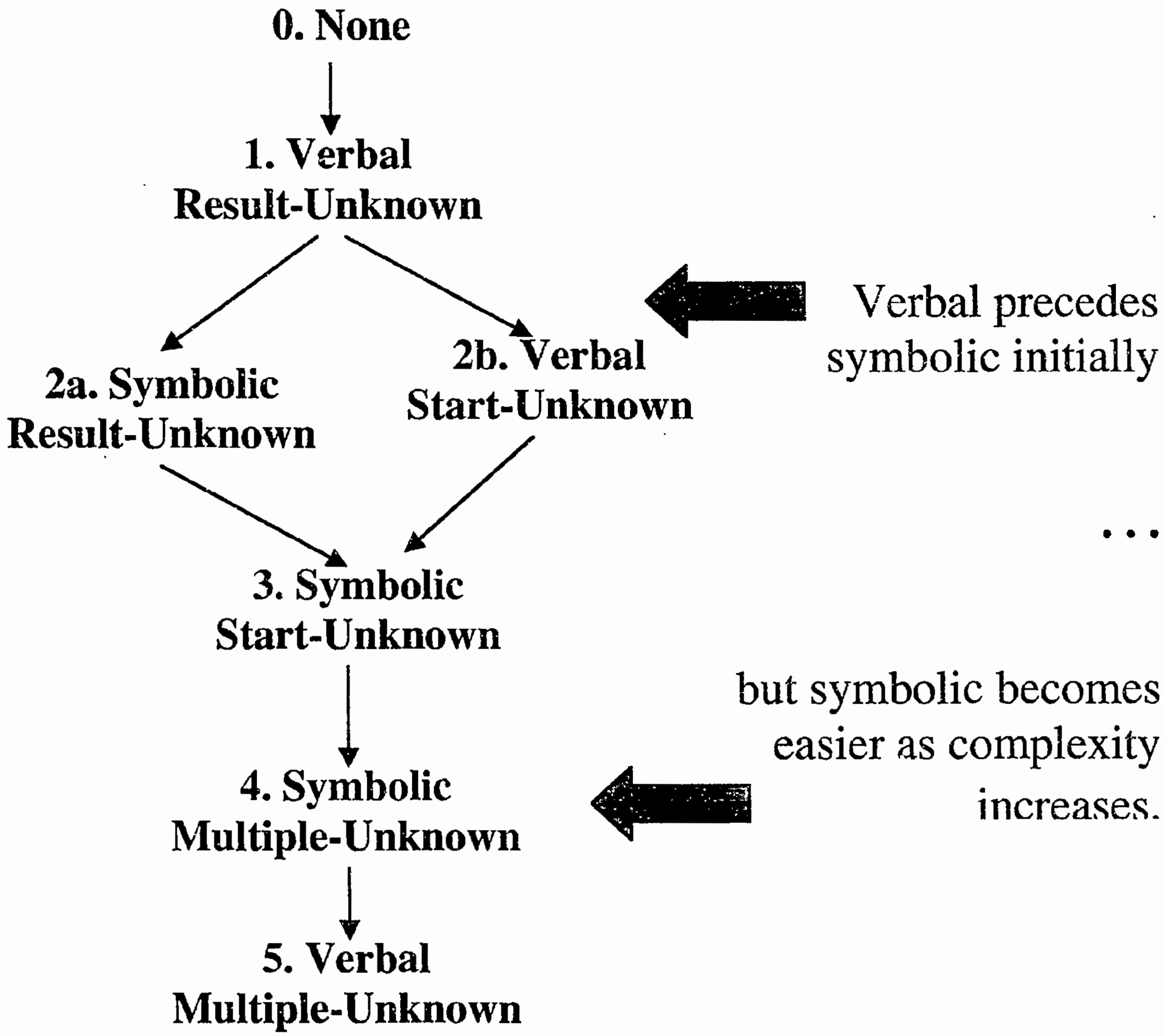


Figure 3. Levels of algebra development.

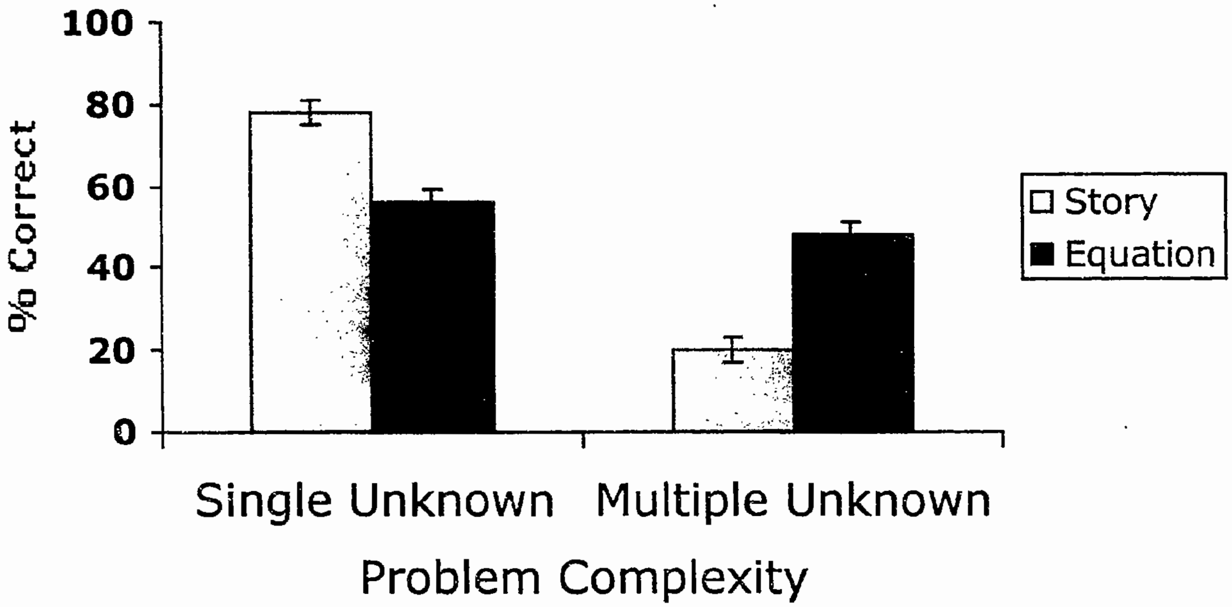
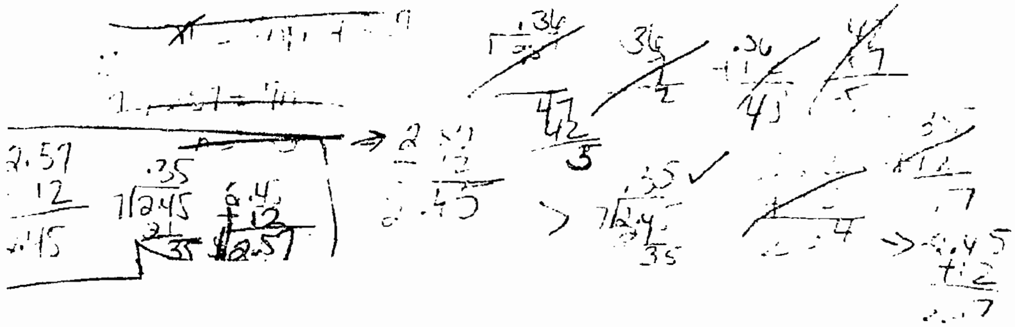


Figure 4. The interaction of problem complexity and representation in Experiment 1.

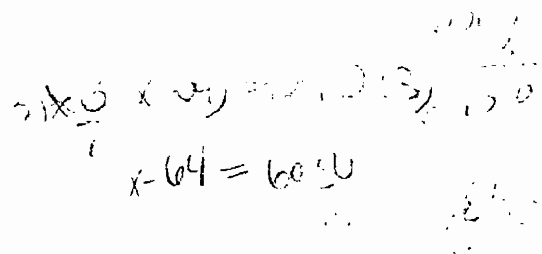
1. After buying donuts at Wholey Donuts, Laura multiplies the price per donut by the 7 donuts she bought. Then she adds the $0.12 charge for the box they came in and gets $2.57. What is the price per donut?



a. Student 87 successfully using unwind on the Donuts start-unknown story problem.

5. Solve for the unknown value, X.

(X - 64) ÷ 3 = 20.50



b. Student 87 (same as in a) incorrectly equation solving on the start-unknown Lottery equation. The error is in the second step where the student subtracts 64 rather than adding 64 to both sides.

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5. After hearing that Mom won a lottery prize, Bill took the amount she won and subtracted the $64 that Mom kept for herself. Then he divided the remaining money among her 3 sons giving each $20.50. How much did Mom win?

$3(20.50) + 64.00$

$61.50 + 64.00 = \boxed{\$125.50}$

c. Student 46 successfully using unwind on the Lottery start-unknown story problem.

1. Solve for the unknown value, X.

7X + .12 = 2.57

$7x + 12 = 257$

$7x = 245$

$x = \frac{245}{7}$

$\boxed{x = 35}$

d. Student 46 (same as in c) incorrectly equation solving on the start-unknown Donuts equation. The error is in the first step where the student appears to multiply both sides by 100 but misses converting 7 to 700.

Figure 5. Two students at developmental level 2b (see Figure 3).

2. Solve for the unknown value, X.

X - 0.15X = 38.24

.85x = 38.24

.85x = 38.24

x = 38.24/.85

x = 44.98

1.00 − .15 = .85

44.98

.85) 38.24

a. Student 19 successfully solves the multiple-unknown equation for the Discount problem.

4. You are in Paris, France and you want to exchange your dollars for French Francs (FF). The first exchange store gives you 5.7 FF per dollar, but charges 22 FF for each exchange. The second exchange store gives you 5.4 FF per dollar, and does not charge a fee. When are the charges from the two stores the same, in other words, what amount of dollars results in the same charge from both stores?

5.4 ~~FF~~ x = 5.7x + 22

− 0.3x = 22FF

x = − 7.33 FF

~~5.7 FF/$ + 22 FF/Exch = 5.4 FF/$~~

~~0.3 FF/$ + 22 FF/Exch~~

~~.3x = 22~~

~~x = −7.3~~

b. Student 19 fails on the multiple-unknown story problem (Exchange story). The student performs an incorrect translation to an equation (the "+ 22" should be added to 5.4x, not 5.7x) and then does not appear to notice that the negative result of equation solving (-7.33) is an unlikely answer to this problem situation.

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6. Solve for the unknown value, X.

5.7X - 22 = 5.4X

57x - 220 = 54x

-220 = -3x

$x = \frac{220}{3} = 73\frac{1}{3}$

c. Student 82 successfully solves the multiple-unknown equation for the Exchange problem.

4. Roseanne just paid $38.24 for new jeans. She got them at a 15% discount. What was the original price?

$38.24

× .15

x + .15 = $38.24

d. Student 82 fails on the Discount story problem. The student attempts both an incorrect informal strategy (multiplying 38.24 by .15 on the left) and an incorrect translation to an equation ("x + .15" should be "x - .15x").

Figure 6. Two students at developmental level 4 (see Figure 3).

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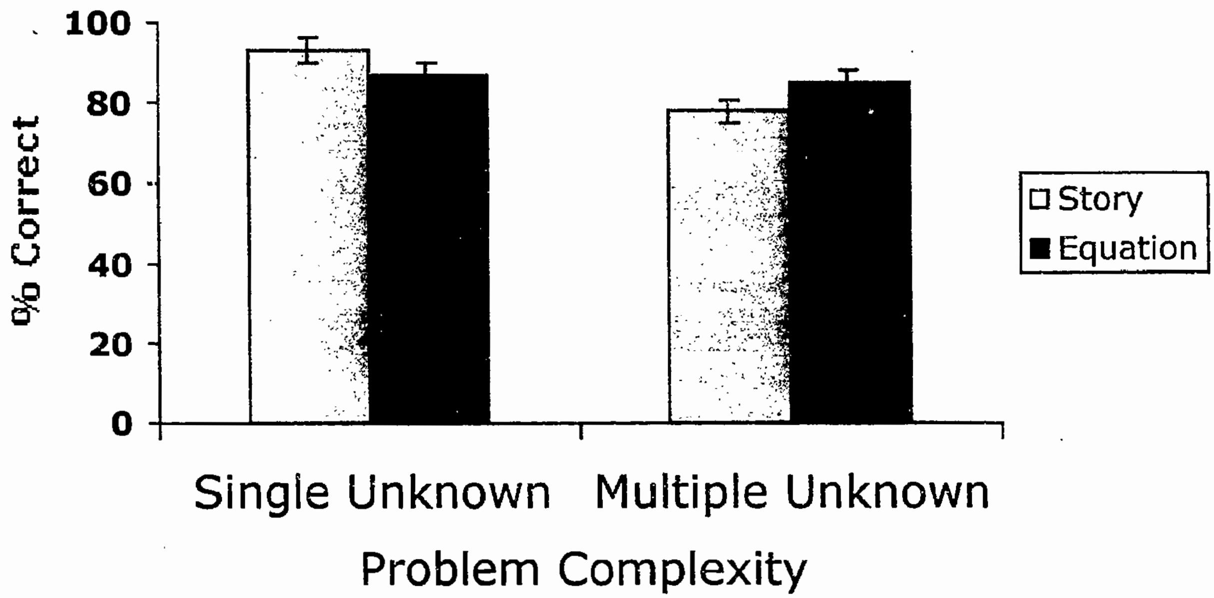

Figure 7. The interaction of problem complexity (number of unknowns) and representation in Experiment 2.