Verbal reasoning as a critical component in early algebra

Kenneth R. Koedinger and Hermina J. M. Tabachneck
Carnegie Mellon University

Presenter: Kenneth R. Koedinger
HCI Institute, SCS
Carnegie Mellon University
Pittsburgh, PA 15213
Email: koedinger@cmu.edu

Recent interest in "early algebra" (Smith, 1994), that is, in teaching algebra in middle school or earlier, has brought up the question: What kinds of activities qualify as algebra or as algebraic thinking? Debates center around whether the use of algebraic symbols and procedures is critical or whether algebraic thinking is something more abstract like the ability to deal with unknowns or induce operational structure (e.g., Kaput's Algebra Working Group electronic mailing list). However, a more fundamental and tractable question can be arrived at by considering algebra, not as an end in itself, but as a means to more effective problem solving[1]. Taking this problem solving perspective, we can reframe the early algebra question as: What kinds of problems challenge the edge of students' pre-algebra competence?

This perspective suggests an instructional focus on moving students forward in their mathematical competence rather than on a concern for whether or not this involves "algebra" per se. This study helps to characterize potential bridges to greater competence by identifying factors that cause early algebra students difficulty. The study compared the effect on students' performance of multiple variants of the same problem solving situation which all share the same mathematical deep structure (e.g., $x * 5 + 34 = y$). The examples below illustrate the dimensions of particular interest.

[1] Here we mean to construe problem solving broadly to include communicating solutions, investigating general relationships, etc.

Result unknown problems:

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Story:  After buying donuts at Wholey Donuts, Laura multiplies the 7 donuts
she bought by their price of $0.37 per donut.  Then she adds the $0.22
charge for the box they came in and gets the total amount she paid.  How
much did she pay?

Word-equation: Starting with 100, if I subtract 40 and then divide by 3, I
get a number.  What is it?
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Equation: 3 * 5 + 34 = x

Start unknown problems:

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Story:  After buying donuts at Wholey Donuts, Laura multiplies the number
of donuts she bought by their price of $0.37 per donut.  Then she adds the
$0.22 charge for the box they came in and gets $2.81.  How many donuts did
she buy?

Word-equation: Starting with some number, if I subtract 40 and then divide
by 3, I get 20.  What number did I start with?

Equation: x * 5 + 34 = 49
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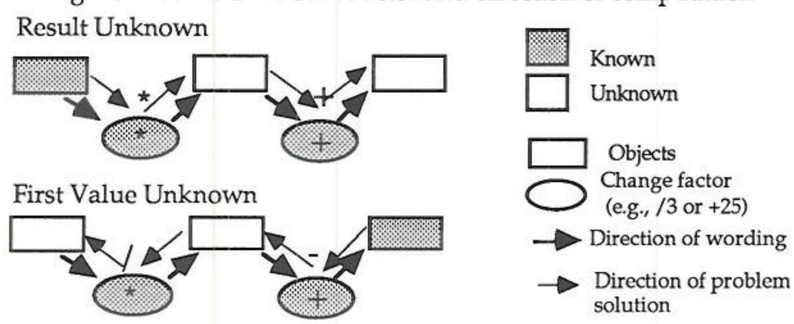
The representational format of the problem statement was one of three types: story, word equation, or equation. The position of the unknown was either at the end of the problem, "result unknown", or at the beginning of the problem, "start unknown". Start-unknown problems require students to compute the unknown by inverting the arithmetic operators given in the problem and applying them in the reverse order (see Fig.1).

Figure 1: Value Unknown Factor and direction of computation



This study is part of a larger set of studies that look at factors involved in the transition between arithmetic and algebraic reasoning (Tabachneck, Koedinger, & Nathan, 1994, Koedinger & Tabachneck, 1994). In those previous studies, we identified a number of non-symbolic strategies in students algebra problem solving and have begun to develop a cognitive model of these strategies, their selection, and how they interact with symbolic algebra. We were particularly interested in students use of verbal reasoning to solve these problems. We designed this study to further explore the importance of verbal reasoning in the development of algebraic competence.

Method

Subjects. Seventy-six 9th grade students participated in the study. Of these students, 58 were enrolled in one of 3 mainstream 9th grade Algebra 1 classes, and 18, who took Algebra 1 in 8th grade, were enrolled in a gifted program Geometry 1 class. The classes were taught by four different teachers. Although the subjects are 9th grade algebra students, they are from an urban public school and are considerably under-prepared. These students scored not far above chance on an Iowa algebra *readiness* exam given at the *end* of the school year.

Materials and Procedure. The study was presented as a quiz during a regularly scheduled class period, and students were given eight problems to solve within 18 minutes. Each quiz consisted of four word problems, two word-equation problems, and two equations; four result-unknown and four start-unknown problems. There were 16 different quiz forms altogether, which, among other things, counterbalanced for order effects, for operations (multiply/add vs.divide/subtract problems) and for difficulty (integers vs. rational numbers). Four different cover-stories were used for the story problems.

Results and Discussion

Representation: There was a highly significant main effect for representation ($F(2,75) = 12.6$, $p < .0001$). Out of a possible score of 4 points, the means and standard deviations for the three groups were: Story-problem (3.0, 1.4), word-equation (2.9, 1.7) and equation (2.1, 1.9). Post-hoc contrast tests show that equations are more difficult than both story problems and word-equations. Surprisingly, there is no difference between story problems and word-equation problems. This suggests that symbolization itself is a main barrier in algebra problem solving rather than, or perhaps in addition to, decontextualization. One might hypothesize that the difference resulted because students attempted to use unfamiliar algebraic manipulations only on the equations while solving the other problem types with verbal reasoning and arithmetic. However, at best students appeared to use a symbolic algebra strategy only slightly more often in the equation problems than in other problems. Further analysis of students' strategy selections and errors is ongoing.

These findings seem to run counter to above mentioned findings in early elementary grades where story problems are significantly more difficult than symbolic problems (Carpenter, Corbitt, Kepner, Linquist, & Reys, 1980; Cummins, Kintsch, Reusser, & Weimer, 1988). One explanation for these seemingly opposite results is that, in first grade, students have had more practice with numeric symbols than with arithmetic verbal language, which, in addition to the arithmetic jargon words, is often considerably more difficult in structure than the language they are (barely) reading, while at high school age, students are considerably more practiced in arithmetic jargon as well as complex verbal language but are lacking practice in properly interpreting algebraic symbols and symbolic forms.

Value unknown: There was also a highly significant main effect for value unknown ($F(1,75) = 48.9$, $p < .0001$). The start unknown problems were significantly more difficult (mean, 2.2; s.d., 1.8) than the result-unknown problems (mean, 3.1; s.d., 1.4). This is analogous to findings from a study by Briars and Larkin (1984) who found that problems of the form ?+A=B were consistently more difficult than problems like A+B=? for lower-grade elementary children in an overview of 11 studies by various authors.

Ability: Both the representation and the value unknown main effects were highly significant for all the mainstream classes. For the gifted group, both effects showed large trends in the right direction, but were only marginally significant perhaps due to the small number of gifted students.

A replication of this study is in progress which will be discussed in the full paper.

Conclusion

The study yielded two important results. First, start unknown problems are more difficult in all forms than result unknown problems. On one hand, students often solve the start unknown problems without resort to symbolic algebra indicating that, in the traditional sense, these problems are not "algebra" problems. On the other hand, because such problems are substantially more difficult even for beginning algebra students they are on path to the development of algebra competence. Such problems use some of the same processes needed for algebra (such as inversion of operators), and if students cannot verbally, qualitatively reason about why operations need to be inverted in a situated problem statement, they can hardly be expected to do algebraic manipulations with understanding. The second important result relates to the manipulation of problem representation. Contrary to previous claims about the difficulties of word problems, we found that students performed better on word problems than on the corresponding symbolic problems. This result indicates that students have the ability to perform "algebraic" operations verbally prior to an ability to do so symbolically.

The focus of this study on students' verbal reasoning strategies helps to characterize their prior knowledge. By building explicitly on students' prior informal knowledge, students can assign meaningful interpretations to newly taught strategies and concepts. This leads to an integrated approach to mathematics instruction where informal verbal knowledge serves to support strategy acquisition and application during the problem-solving process.

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