# 13

# Conjecturing and Argumentation in High-School Geometry Students

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There is a tendency to think of individuals who can discover new ideas or develop convincing arguments as having special "talent" or superior "intelligence." This view of conjecturing and argumentation abilities as fixed traits suggests that instruction directed toward such abilities is pointless for all but the most gifted of students. In direct contrast to that view, this chapter argues that successful conjecturing and argumentation performances are the consequence of particular skills and knowledge. In an appropriately structured learning environment, such skills can be acquired by anyone.

One reason for doubt regarding the instructability of conjecturing skills is our limited understanding of what these skills are. Developing a model of these skills is a key step toward creating effective learning environments for conjecturing. This model can then provide design guidance in creating elements of a learning environment: conjecturing tasks that appropriately challenge students and forms of assistance (including manipulatives, facilitative talk, and computer software) that support student learning. The learning approach being advocated here has important similarities with the Vygotskian notion of assisted performance (Vygotsky, 1978) and more recent variations like cognitive apprenticeship (Collins, Brown, & Newman, 1989).

This chapter presents a cognitive analysis of student conjecturing that includes a task analysis and an initial model of conjecturing, observations of student performance in a dynamic assessment, and modifications to the proposed model as guided by the results of this assessment. This cognitive analysis is then used to suggest forms of student assistance, including computer software tools, activities that draw out conjecturing skills, and facilitative "talking points" (hints or prompts), to help students through the most difficult terrain on their ways to conjecturing skill.

## A TASK ANALYSIS AND PRELIMINARY MODEL

The cognitive analysis of conjecturing focuses on a representative conjecturing task called the Kite task, an adaptation of one that appears at the end of *Discovering Geometry* (Serra, 1989):

*Kite Task—Part 1.*

> A "kite" is a special kind of quadrilateral whose four sides form two pairs of congruent adjacent segments. In other words, a kite is a quadrilateral ABCD with AB congruent to CB and AD congruent to CD.
>
> Investigate these figures called kites using whatever tools you would like and discover and write down what must be true of every kite.

*Kite Task—Part 2.*

> Here is a statement that a student in another class made: "The diagonals of every kite bisect each other." Do you believe it? Why or why not?

The Kite task was designed as a dynamic assessment whereby students are given a difficult open-ended problem in the presence of an interviewer who plays the combined roles of assessor and tutor. The interviewer was prepared with a set of prompts or hints that could be cautiously given to students when they showed clear signs of being stuck. These prompts were organized in a hierarchy of goals created through a rational task analysis of the demands of the conjecturing task. The Kite interviewer's *only-as-needed* presentation of prompts ensures that we can see some signs of success from all students.

### A Preliminary Task Analysis of the Goals of Conjecturing

As a first step toward understanding the goals of conjecturing and argumentation more generally, we performed a task analysis (Newell & Simon, 1972) to identify the major goals involved in performing the Kite task:

1. Draw an example of a kite.
2. Draw an example that is not overly specific (not a rhombus).
3. Make a conjecture.
4. Prove a conjecture.
5. Find a counterexample to reject a false conjecture.

We considered what knowledge was needed to achieve each goal and, in particular, in what ways students were likely to have trouble. The output of this analysis was a list of hints and prompts within each goal. These hints were the basis for our dynamic assessment. The initial hints within each goal are vague; subsequent hints get successively more specific. This approach has proven successful in our previous work on intelli-



gent computer-based tutors (Anderson, Corbett, Koedinger, & Pelletier, 1995).

These goals and hints reflect our initial hypotheses about the nature of students' conjecturing and argumentation skills. One hypothesis was that getting started on this open-ended problem would be difficult for students who are used to 1-min. problems with a single right answer (Schoenfeld, 1989). Thus, the hints for the first goal ("Draw an example of a kite") prompt the student toward getting started: "Think about what a kite looks like" and then "Why don't you draw an example of a kite?"

A second hypothesis was that students were likely to create and investigate overly specific instances of kites, in particular, drawing a rhombus rather than a kite. Hints suggesting that students "Draw some more kites" and, more explicitly, that "Your kites all have four congruent angles" address this concern.

Third, we thought that focusing students' attention on individual geometric objects and on measuring these objects would help students make conjectures. Thus, within the third goal are hints like "Look at the angles in a kite" and "Use your protractor."

Fourth, we hypothesized that students might not spontaneously provide evidence for their conjectures and thus were prepared with the prompts "How do you know the things you wrote down are always true?" and "How would you convince someone else?" Further, we thought students' arguments might progress in sophistication from statements of self-evidence, like "I just know," through reference to a single supporting example, then to multiple examples, and finally to a deductive argument.

Fifth, based on the results of Senk (1983) we knew that even for students who were successfully writing geometry proofs in class, formulating a proof problem from a conjecture was likely to cause considerable difficulty. Thus, we included the subgoals "4.2.1 Identify a reference diagram," "4.2.2 Identify the goal statement," and "4.2.3 Identify the given statements" and provided a substantial number of prompts within each like "What do you know about kites? That should be your given."

Sixth, we thought that a substantial number of students would not generate a false conjecture and therefore not need spontaneously to engage in conjecture-defeating argumentation. Thus, we added a conjecture evaluation task (Part 2) to address the goal "Find a counterexample to reject a false conjecture."

## THE QUALITATIVE STUDY

### Method

About 60 students were interviewed at two sites in the Pittsburgh Public Schools. One school was using the *Discovering Geometry* textbook (Serra, 1989) with its emphasis on introducing students to geometry through inductive investigations often done in collaborative groups. The other school was using a traditional geometry textbook with its emphasis on introducing definitions and postulates, proving and applying theorems. About half the students at each site got a variation of the Kite task where we changed the term "kite" to "tike." Given the everyday meaning of the term kite, which is suggestive of a particular form, we decided to see what effect the term had on student behavior by comparing against a neutral term "tike."

Each of two interviewers performed two interviews during a 44-min. class period. Interview time was about 20 min. Students were randomly selected from the geometry classes and came to a separate, quiet office for the interview. The interviews were performed in late May at the end of a year of geometry instruction. During the interview, students were provided with the following tools: compass, ruler, protractor, pencil, and as much paper as they needed. Students were asked to think out loud, and we prompted them to "keep talking" whenever they fell silent (Ericsson & Simon, 1984).

*Procedure.* The interviewer began by reading the statement:

> [Student name], we are interested in how geometry students think about problems. I am going to give you a problem to work on and while you are working on the problem, I'd like you to think out loud. In other words, I'd like you to say what you are thinking as you are thinking it. You don't have to worry about whether everything you say is right or wrong, just tell me what you are thinking. Does that make sense? [Pause and answer any student question.] If you don't mind, we are going to record this. Of course, this data will be confidential and we will not use your name when we refer to it. Do you mind if we record this? [Pause and answer any questions.]
>
> Occasionally I may say "keep talking" if you become silent. Please talk loudly and clearly. [Start the tape.] For the tape, this is student ID number [student's ID number]. Here's the problem [give student the problem] and there's extra paper if you need it. Start by reading the problem out loud and then keep talking. [Start timing]

The student then read the problem statement, "Draw an example of a kite," and was asked to think aloud as he or she worked. Our goal was for students to do as much independent work as possible. However, when the interviewer judged that student was at an impasse, he or she would give one hint as guided by the interviewer form. Signs of a student impasse included long pauses, statements of confusion or frustration, or off-task behavior.

The interviewer checked off hints as they were given and, if necessary, indicated additional hints in the blanks provided. As much as possible, hints were chosen sequentially from the earliest goal not yet achieved and, within each goal, from the earliest hint not yet checked off. Students, however, would sometimes begin to pursue a "later goal" before completing an "earlier one," for example, beginning to make conjectures (goal 3) before having drawn a kite that is not overly specific (goal 2). If hints were needed in such cases, the interviewers would follow through on the student's goal if it made sense (e.g., if the student's difficulty was in how to achieve this goal). If, instead, the student wasn't sure what to do next, the interviewer would choose a prompt from the earliest goal that the student had not yet achieved. In general, the interviewers put priority on coherent interaction over strict adherence to the interview form.



The initial hints within each goal were vague, but further hints were progressively more detailed and directive. The interviewer treated each goal independently. A detailed hint might be provided on one goal, but on the next goal the student was left on his or her own unless the student showed clear evidence of another impasse.

The original plan was for interviewers to pace their hints such that students achieved goal 3 within 10 min. and goal 4 within 15 min. After achieving goal 4, the student was to be given Part 2 of the Kite Task, the conjecture evaluation question. We found this schedule difficult to maintain. Further, we soon recognized that most students were making false conjectures spontaneously; that is, goal 5 was being addressed in the context of the other four goals. Thus, the pacing designed to ensure an opportunity for students to evaluate a false conjecture proved unnecessary. We relaxed the timing constraints and, unless a student was going particularly fast, left off Part 2 in favor of giving students more time on the earlier goals.

## Overall Results

*Curriculum Comparison.* The comparison between students using the *Discovering Geometry* text and those using a traditional text showed no substantial difference. Although it remains possible that a more complete and detailed quantitative analysis might yield some differences, we did not see the kind of qualitative differences that might be expected from the nontraditional approach of *Discovering Geometry*. This result should not be interpreted as critical of this particular text, as at least three mediating factors reduced the likelihood of an effect: (a) greater teacher experience using the traditional text, (b) great variability in the way different teachers implement the *Discovering Geometry* curriculum, and (c) high variability and generally poor preparation of this urban student population. This result *should* be considered as evidence for substantial difficulties in implementing curriculum reform in a way that yields substantial student achievement gains. It takes much more than a textbook.

*Kite–Tike Manipulation.* The Kite versus Tike manipulation showed a large difference in the initial example drawing (goals 1 and 2). Although both groups were equally likely to draw overly specific diagrams (discussed later), the kinds of diagrams drawn were distinctly different in the two groups. Figure 13.1 shows examples of the initial diagrams typical of the Kite and Tike groups. The Kite group (see Fig. 13.1a) drew figures that looked like rhombi with slanted sides such that the diagonals, if drawn, would be horizontal and vertical. The Tike group (see Fig. 13.1b), on the other hand, drew figures that looked like squares (sometimes rectangles

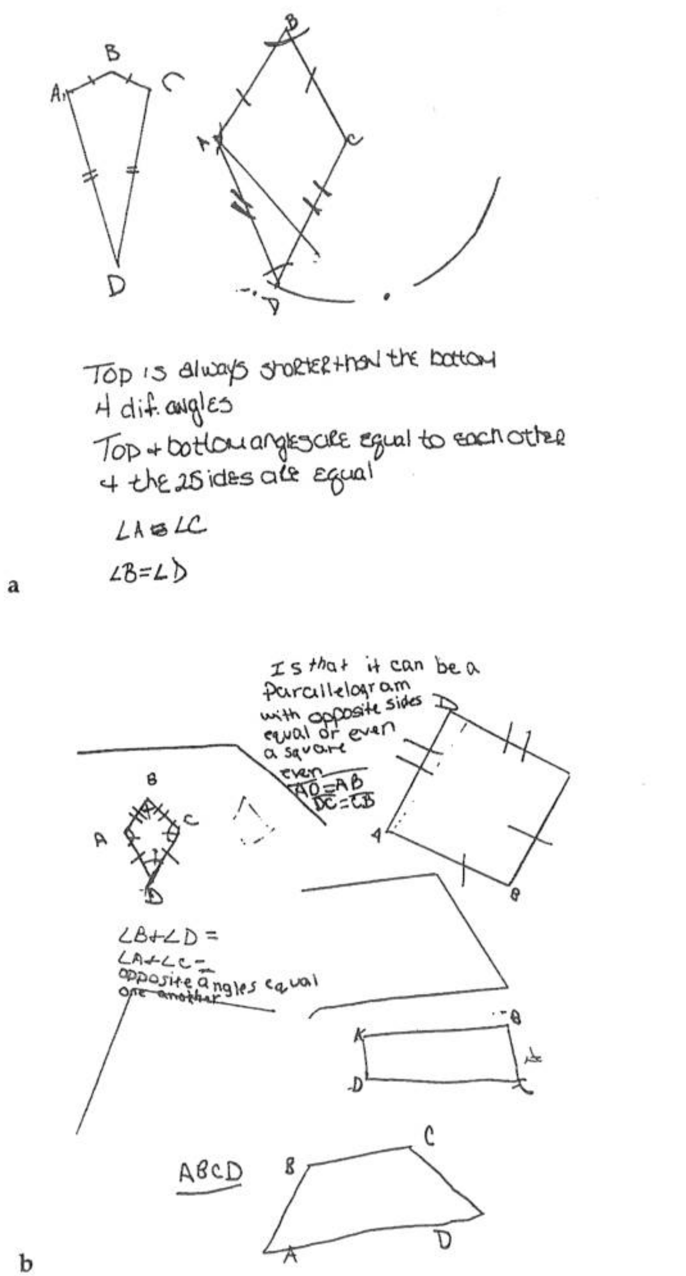



FIG. 13.1 (a) A sample of student work on the Kite task. (b) A sample of student work on the "Tike" version of the Kite task.



initially) with the sides drawn horizontally and vertically and the diagonals slanted.

Although quite strong, these differences were limited to the way the figures were drawn and had little bearing on students' success. The Kite condition might be expected to yield more fluent conjecturing because the specialized world knowledge cued by "kite" might help generate ideas. However, this expectation was not borne out. Similarly, there were no signs of an effect of the Kite–Tike manipulation on conjecture evaluation (goal 4).

## Qualitative Analysis

We observed a wide range of performance in these urban high-school students. At one extreme, we saw that there was plenty of room for improvement in conjecturing skill—many students had difficulty right from the start in simply creating an initial example to investigate (drawing an example kite). At the other extreme, we saw a few students (2 or 3 out of 60) exhibit a full range of conjecturing skills as they successfully performed the task with little or no assistance, for example, by conjecturing and proving that the diagonals of a kite are perpendicular. In between these extremes, the dynamic assessment method allowed us to observe a wide variety of skill levels across students and across goals. Because we provided assistance only as needed, some students lacking skill in one aspect of conjecturing (e.g., example creation) were able to exhibit skill in another (e.g., conjecture induction). Although most students would have gotten practically nowhere on their own, almost all students, with the support of occasional prompts, were actively engaged in this task. Each student made reasonable progress, and many showed glimmers of talk and reasoning characteristic of mathematicians.

What follows is a descriptive analysis of the variety of student performances within each of the major goals. The focus is to provide a qualitative characterization of what conjecturing skills were or were not exhibited by students.

*Goals 1 and 2: Drawing an Example That Is Not Overly Specific.* As we anticipated, many students had difficulty getting started. Twenty-five percent of the students needed to be prompted before they began drawing an example kite. Students had even more difficulty in drawing a kite that was not overly specific. Only 15% did so without prompting, another 35% were able to do so on their own after some prompting, and a full 50% needed to be shown how to do so. Many of students' overly specific initial diagrams were the result of the same common pattern of behavior. Such students would first draw two sides of equal length that shared a vertex (Fig. 13.2a). When drawing a third side, they would invariably begin to draw it parallel to an existing side (Fig. 13.2b). The consequence is that when the student tried to make a fourth side equal to the third side, they were forced to create a rhombus. (If $1 = 2$ and $2 \parallel 3$ and $3 = 4$, then the figure is a rhombus.) Students' tendency to draw a rhombus appeared to re-

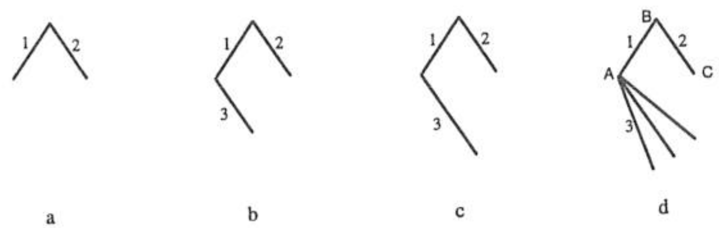



FIG. 13.2 A typical student's attempt at constructing a kite. (a) Sides 1 and 2 are drawn equal in length. (b) Side 3 tends to be drawn parallel to side 2 and thus, to make sides 3 and 4 equal in length, both sides 3 and 4 end up equal in length to sides 1 and 2. (c) With prompting to make sides 3 and 4 different lengths from those of 1 and 2, the student reaches an impasse. (d) The impasse is resolved by swinging the ruler to find a point where sides 3 and 4 intersect and are equal in length to each other, but not to sides 1 and 2.

sult less from a preconceived notion of what the complete figure should look like, than from a bias to draw parallel lines.

When prompted to draw a kite where sides 1 and 2 were different from sides 3 and 4, the following behavior of one student provides an illustration of the way many students struggled. This student picked 2 in. as the length of sides 1 and 2, and then decided that sides 3 and 4 should be 3 in. long. When he drew a longer side 3 (again parallel to side 2), he reached an impasse (see Fig. 13.2c). Putting his ruler between the unattached end points of sides 2 and 3, he saw that side 4 would not be 3 in.

At this point many students decided it wasn't possible to make sides 3 and 4 different from sides 1 and 2. But this student continued to experiment trying different positions for side 3 (Fig. 13.2d). Some found the right spot with such discrete experiments, but this student decided that he could swing his ruler, fixed at the intersection with side 1, to mark off all the possible points 3 inches from point A. Then he performed the same ruler-sweeping procedure from point C and so found the point at which sides 3 and 4 would each be 3 in. long. Indicative of how limited to particular temporal and situational contexts knowledge can sometimes be, this student reinvented the standard compass construction procedure (using his ruler) without recalling he had been taught this procedure the prior fall *and* even though there was a compass in the tool set in front of him.

*Goal 3: Making a Conjecture.* Students did not have much trouble in coming up with *some* conjecture, that is, in saying something they thought would be true of every kite. However, many of these conjectures were trivial, essentially repetitions of the givens (e.g., a kite "has four sides" or "has AB = BC and CD = AD"). Most students were able to make a meaningful, nontrivial conjecture, either on their own or with some prompting. However, 25% were unable to do so within 20 min. These students typically had much difficulty in the drawing phase and were weak in their knowledge of basic geometric concepts and notations (e.g., angles and how to label them).



Although we expected some students to draw overspecialized figures and, thus, be in a position to make false conjectures, we were surprised to see how common such conjectures were. The most frequent false conjecture was "$\angle B = \angle D$." Students who drew rhombi correctly observed that these angles looked equal in their overspecialized version of a kite. Some other false conjectures were also consequences of drawing a rhombus (e.g., "All the 'little triangles' are congruent" from a student who had drawn the diagonals).

Other conjectures, like "$\angle EAD = 60°$," seemed to have their source in the particular examples the student drew or as variation of a geometry theorem the student recalled (e.g., an equilateral triangle has 60° angles). This second kind of memory-based influence on conjecture generation is notable. It is not a form of inductive reasoning in the usual sense of generalizing from instances, nor is it deductive, deriving logically from existing postulates or theorems.

*Goal 4: Prove a Conjecture.* Almost all students seemed satisfied to stop after making one or a few conjectures from the example(s) they had drawn. Only a rare few showed any unprompted signs of thinking that further evidence was necessary or desirable (goal 4.1), and, in at least one such case, the student appeared to provide a proof more out of adherence to classroom habit than out of a self-motivated desire to validate his conjecture. More often, students seemed confused when asked for further evidence, as in the first hint for goal 4.1: "How do you know the things you wrote down are always true?" The next prompt, "How would you convince someone else?," was considerably more effective in eliciting an attempt to provide some justification.

Many students passed through the progression of first saying the example they had drawn was enough evidence and then, with prompting, that many positive examples are sufficient evidence. Most students had to be asked explicitly to write a proof "like you do in class" before they made such an attempt.

Students had clear difficulty in setting up proofs for their conjectures (goal 4.2). Even though they had been exposed to many proofs during the school year, they had difficulty expressing a conjecture in terms of the "given statements" and the "goal statement." Students' conjectures were usually stated just as a conclusion (e.g., "Angles A and C are always equal") without stating the premise (e.g., "In kite ABCD, angles A and C are always equal"). Accordingly, students found it easier to specify the goal of the proof problem, but establishing the givens, that ABCD is a quadrilateral with AB = BC and AD = DC, was more difficult. Few students spontaneously referred to or drew and labeled specific figures as reference diagrams for the proofs.

Few students (about 10%) successfully formulated proof problems and began to work on them. One of the big difficulties in coming up with a proof (goal 4.3) was adding the segment BD (a "construction") to create congruent triangles. Even when told to draw this segment, a number of students needed quite specific hints in order to finish the proof (e.g., hints 4 and 5 in goal 4.3.1: "What methods do you know for proving [congruent triangles]?" and "Do you remember [Side–Side–Side]?").

At the other extreme, two students with little or no prompting were able to conjecture and prove that the diagonals of a kite are perpendicular. Interestingly, while working toward a proof of this conjecture, one student deduced a new conjecture she hadn't previously thought of. Although the discovery of a conjecture is usually the product of induction from examples, this student's work illustrates that proof itself can also serve as a discovery tool.

### Summary of Observations

We made the following observations of students' performance on this conjecturing task:

1. Experimenting with a class of figures by constructing and examining examples is a difficult but significant skill for students.
2. Example construction is strongly biased by subtle perceptual (e.g., parallel-lines bias) and linguistic (e.g., analogical suggestiveness of the label *kite*) influences. It is difficult for students to break out of the "set" caused by these influences.
3. Generating conjectures is not difficult per se; however, many conjectures students generated were either trivial repetitions of the problem conditions (e.g., AB = CB), not particular to kites (e.g., has 4 sides), or false inferences from overspecialized examples (e.g., $\angle B = \angle D$).
4. Students have difficulty differentiating claims (conjectures) and evidence for those claims (argument). When an argument is elicited, it is much more likely to be in the form of failure to falsify, "Every time I tried it, it worked," than in the form of a deductive proof.
5. Formally stating conjectures is difficult.
6. Deductive proof can lead to new conjectures.

## TOWARD A COGNITIVE MODEL OF CONJECTURING

### Framework for a Conjecture Model

Figure 13.3 shows the hypothesized components of a cognitive model of conjecturing, organized in a goal-structure diagram. As indicated by the top goal, conjecturing skills are relevant not only for discovery, but also for problem solving and recall. The two major subgoals are Generate Conjecture and Argue For Or Against. This goal-structure depiction indicates the major component processes in an approximate ordering. The ordering, however, is only suggestive. These processes or strategies are not executed sequentially by students, by mathematicians, or by scientists; instead, they are opportunistically applied based on specific problem demands. It is also not the case that each subgoal has a unique process for achieving it. A particular subgoal may have multiple processes or strategies for achieving it. For example, to Generate Conjecture one can Analogize, Investigate, or Deduce. Conversely, a particular strategy may be useful for multiple subgoals; for example, Investigate can be used to Generate Conjecture or to Find Examples For or Find Counterexamples Against.



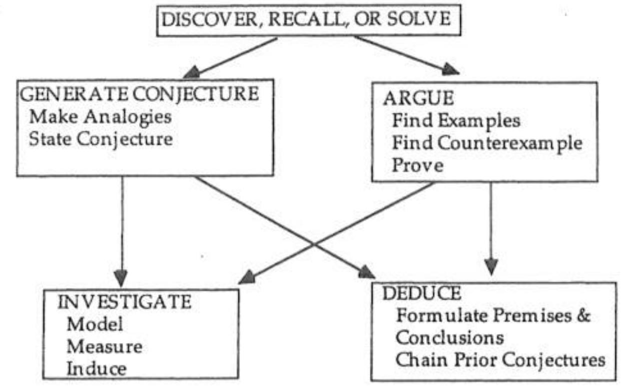



FIG. 13.3 The goal structure for conjecturing and argumentation skills.

There is a temptation to associate empirical investigation with conjecture generation and deduction with conjecture evaluation; however, evidence presented here and elsewhere suggests otherwise. Using terminology more common to science, Klahr and Dunbar (1988) presented a model of scientific discovery in which "experimentation" (a kind of Investigate) is relevant for both generating hypotheses (Conjectures) and for testing hypotheses (Examples For and Counterexamples Against). Chazan (1993) documented "student preference for empirical arguments over deductive arguments when presented with mathematical problems." In Chazan's interviews with high-school students, he found students were more likely to view empirical evidence as "proof" than they were to see deductive proof as such.

The role of deduction in conjecture generation was not recognized in these prior works. The possibility of deductively generated conjectures is clear from rational grounds. Postulates and theorems can be used to chain forward from the given premises of the investigation (e.g., In kite ABCD, AB = BC and AD = CD) toward new conclusions (ones not previously generated by other means). When these processes are used to aid conjecture recall or rediscovery, deduction provides an important avenue to reconstruction of forgotten conjectures. The opportunistic and sometimes redundant application of alternative strategies to achieve a variety of goals is a crucial aspect of mathematical understanding (Koedinger & Anderson, 1991; Tabachneck, Koedinger, & Nathan, 1994).

### Conjecture Generation

Conjecture generation includes generating an idea for a conjecture, making sure it is "interesting," and making sure it is at least consistent with the examples that are being investigated. The simplest filter on the interest of a conjecture is that it at least say something new, something beyond repeating the premise. A number of students in the Kite study made trivial conjectures ("Kites are quadrilaterals" or "AB = BC and AD = BD") that simply repeated an aspect of the definition. It may be that students are unable to distinguish between such trivial conjectures and potentially more interesting ones, but it seems more likely that students were unable to access more sophisticated strategies for conjecture generation than simply repeating statements from the text.

A slightly more sophisticated strategy, although still quite weak, is to reason by analogy, using similar prior knowledge. We saw this approach in the Kite study when students conjectured the conclusion "$\angle ADC = 60°$" by analogy to equilateral triangles. Klahr and Dunbar (1988) also found analogy to be an important source for hypothesis generation in scientific discovery. The analysis here, however, focuses on two more-systematic approaches for conjecture generation, Investigate and Deduce.

## Investigation Skills

One method for generating a conjecture involves investigating examples and inducing any relationships that appear invariant. In the Kite task, students drew examples of kites, inspected and sometimes measured them, and noticed relationships (e.g., the two sets of opposite angles look congruent, or the diagonals look perpendicular). More generally, to Investigate, one must Model, Measure, and Induce (see Fig. 13.3).

*Model Creation.* One way scientists begin an investigation is to explore examples or models in some general area of interest, like falling objects (to study gravity) or the *Escherichia coli* bacteria (to study germ propagation). An initial model can be created using that part of the conjecture that comes from the research question, the premise in a premise-driven investigation, or the conclusion in a conclusion-driven investigation. To investigate "If a figure is a kite, then X," we create diagrammatic models of kites and explore their characteristics. To investigate "If X, then success in college is more likely," we find models of college success (college graduates) and explore their characteristics. In both cases, it is also important to explore nonexamples (quadrilaterals that are not kites or students that didn't graduate) to avoid conjectures that apply more generally ("Kites have four sides" or "Students who can read are more likely to complete college").

An important general heuristic for modeling is to create or find models that are not overspecialized (i.e., having more features than is necessary). For instance, we don't use a quadrilateral with all sides equal to explore kites; we don't use doctoral students to explore characteristics of undergraduate success. Related to this "general model" heuristic is an "extreme model" heuristic, which suggests investigating extreme cases, like a concave kite (where D appears "inside" the figure). Most students in the Kite study did not invoke these heuristics. Many of their models were either overspecialized (e.g., all sides equal, parallel sides, or right angles) or prototypical (e.g., horizontal–vertical orientation, convex, average "thickness").



In many domains, like geometry, it is typical for researchers to construct models to investigate. To solve all but the simplest construction problems requires the following problem-solving steps: (a) decompose the problem into solvable subproblems, (b) compose and integrate the solutions to these subproblems, and (c) manage the inevitable interactions between subproblem solutions. For instance, in the Kite task where the construction problem is to create a quadrilateral ABCD with AB = BC and AD = CD, successful students decomposed the task into simpler problems of constructing segments. The subproblem of constructing segment AB equal to segment BC is solved with little trouble. The typical student used a ruler to draw two equal-length segments, say 2 in., that shared point B. The third and fourth segments, AD and CD, present an analogous subproblem and students typically solved it in the same way, which resulted in segments with symmetric orientations and the same lengths (i.e. , AD and CD are also 2 in. long). Applying the general-model heuristic adds the constraint that the pairs of segments should not be equal. Thus, the new subproblem is to make AD and CD, say, 3 in. long. By itself, this subproblem is as easy as the first (make AB and BC each 2 in. long); however, managing the interaction between the two subproblems, that points A and C must be shared, is difficult (as was illustrated earlier in this chapter).

Model construction is sometimes taught in science classes as part of instruction on the "scientific method" or, more specifically but less commonly, instruction on "experimental design." In geometry instruction, the methodology for model construction is typically compass and straight-edge construction. Traditional instruction has so isolated and overrefined this method, however, that its function in empirical investigation is lost on most students and teachers alike. Despite weeks of instruction on compass and straight-edge constructions, students in the Kite study did not employ this method when it was appropriate. However, students are capable of less formal approaches to model construction, including freehand drawing and the use of other measurement and drawing tools. Most students in the Kite study were able to create adequate diagrammatic models, but did so using the ruler and protractor.

By not presenting construction within the context of its use in investigations, traditional geometry instruction leads to student knowledge structures that tie construction methods only to teacher-imposed goals. Thus, students are not likely to access construction knowledge when it is really needed to aid recall or help solve a problem.

*Measurement.* In premise-driven investigation, generating a conjecture amounts to finding a conclusion that follows from the given premises. The focus of measurement is on possible conclusions. The student can measure such things as segment lengths, angles, and areas. Looking for constants or identities among such measures is a good way to find many interesting results. Although there are results that are difficult to rediscover empirically (e.g., the Pythagorean theorem), many conjectures in high-school geometry are "rediscoverable" through straightforward measurement strategies.

Measurement strategies aid in the generation of conjectures that are at least consistent with the examples that have been constructed. In the Kite investigation, many students made the conjecture that angles A and C were equal as the result of focusing their attention on these angles and noticing that their measures looked the same. Having started with an overly specific model (a rhombus), many students also made the consistent but false conjecture that angles B and D are equal. The investigation process of noticing equivalence in measurements is the same for both conjectures.

*Induction.* Koedinger and Anderson (1990) presented a detailed cognitive model of inductive competence in geometry as a key component of a model proof planning. We found that despite years of contact with geometric theorems, high-school geometry experts do not plan proofs by using heuristics to search iteratively through the space of possible theorem applications, as was previously thought (e.g., Anderson, Greeno, Kline, & Neves, 1981; Gelernter, 1963). Instead, they initially make an abstract proof plan by using perceptually based knowledge to lay out the key proof steps, leaving theorem application to fill in the details. They employ a type of inductive reasoning to do so.

A key component of this planning ability is also critical to conjecturing, namely, the use of perceptual knowledge of prototypical diagram configurations to "parse" diagrams into useful parts (e.g., triangle pairs, triangles, angles, and segments). This perceptual configuration knowledge provides a source of guidance in conjecture generation. It cues what parts of a diagram to look at, compare, and measure. Once a student sees a diagram in terms of relevant parts, how they overlap and interconnect, it is not difficult to go the next step and inquire what parts might have invariant properties alone or relative to other parts.

*Using "Model and Measure" in Problem Solving.* Investigation skills are relevant not only for conjecturing, but also for problem solving. A "model and measure" strategy can be used to solve problems without recourse to the deductive application of theorems. Thus, it is possible to use this strategy to solve problems prior to having learned the normative technique for doing so. Consider a high-school geometry student who does not know trigonometry and is faced with the following problem. He works for a small company building customized stereo speaker boxes and needs a method to figure out the length of boards needed to create a slanted roof on the top of the box. He knows the angle of the slant (20°) and the horizontal distance (10 in.). If he knew trigonometry, he could find the length of the top by solving "$\cos 20° = 10/x$" for $x$. Alternatively, the model-and-measure strategy can be used (see Fig. 13.4). Draw a scale model of the situation (1 cm = 1 in.) so that it fits the givens of the problem: that is, the width of the box is 10 cm and the angle of the slant (CAB) is 20°. Measure the desired quantity, in this case segment AC, which is 10.6 cm, and apply the scale factor. The top should be about 10.6 in.

Although this strategy sacrifices some precision and takes more time to perform than the deductive application of a theorem, it is quite sufficient for many problem situations. More importantly, the model-and-measure strategy has three advantages over theorem application. First, it is more general—a student doesn't need to have learned a theorem to apply it. It can be applied, for example, to "trigonometry" problems prior to having learned trigonometry theorems. Second, it is easier to learn and apply than most theorems. For example, the model-and-measure alternative to the Pythagorean theorem does not require algebraic computations. Third and most importantly, the model-and-measure strategy can play a sense-making role in helping students learn new theorems, recall old theorems, and check solutions generated using those theorems.



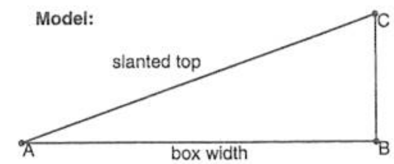



FIG. 13.4 Applying the "model and measure" strategy in problem solving. The goal is to find the length of the slanted top (AC) of a speaker box given the box width (AB = 10) and the angle of the slant (∠CAB = 20°). The strategy does not require trigonometry: draw a scale model, measure the desired length, and apply the scale factor.

## Stating Conjectures

Articulating generalizations is difficult. Students can reason generally about unknowns prior to being able to articulate the generalization with which they are working (Koedinger & Anderson, 1996; Nathan, Koedinger, & Tabachneck, 1996). To do so requires self-reflection and extra cognitive resources that come only from practice. In the Kite task, students' initial conjectures tended to be stated simply as conclusions, with the premise presumed. For example, they didn't say "In kite ABCD, ∠ADC = 60°" but simply "∠ADC = 60°"; not "Kites have four equal sides," but "four equal sides"; not "Opposite angles of kites are equal," but "opposite angles are equal." By assuming the premise and essentially forgetting about it, students are better able to focus their cognitive resources on what they need to measure and generate. However, once a likely conclusion has been discovered, the conjecture does need to be stated in full.

The preceding examples illustrate two different ways of stating conjectures: one, more symbolic, referring to point labels, and one, more natural. The use of natural language tends to express better the generality of the conjecture, whereas the symbolic form may contribute to some students' difficulty in distinguishing evidence and claim, namely, in thinking that the conjecture is about a particular figure rather than about kites in general (see Chazan, 1993). On the other hand, the natural-language form can become cumbersome and difficult to interpret. For instance, attempts to find a natural-language form for "In kite ABCD with AB = BC and AD = CD, angle A is congruent to angle C" can lead to jargon-laden statements like "The opposite angles formed by noncongruent adjacent sides of a kite are congruent."

### Argumentation

Argumentation involves making empirical or deductive arguments for or against a conjecture. Mathematicians commonly convince themselves of a conjecture by searching for a counterexample—if they do not find one after significant time and effort, this itself is reasonable evidence for the truth of the conjecture, particularly when widely varying positive examples can be demonstrated. Extended failure to find a counterexample has been the key source of argument for the conjecture referred to as Fermat's Last "Theorem." There is a recent well-substantiated claim of a proof of this conjecture, but over the years the effort to find a proof has been less justified by a need to convince us of its truth and more justified as a way to facilitate the generation of new mathematical ideas and conjectures. Mathematicians also use deduction to aid conjecture generation. In reasoning outside of mathematics (and perhaps even within it), empirical arguments are much more common than deductive ones (Kuhn, 1991).

In the Kite task, students rarely provided an argument for their conjectures without some prompting. As Kuhn (1991) found, many students appear to lack epistemic knowledge that distinguishes between claims and evidence and, further, lack an appreciation for the value of evidence in convincing others. A key step in the development of argumentation skill is the recognition of the importance of multiple examples as providing better evidence than a single example. Testing a conjecture with multiple examples involves the same Investigate skills—modeling and measuring—discussed previously. In the Kite task, students often referred back to the instances they used in conjecture generation. When pushed for a convincing argument, it was typical for them to invoke measurement (Chazan, 1993).

Few students spontaneously attempted a deductive proof. When prompted to do so, most had trouble formulating their conjectures as proof problems. The skills required for proof formulation are similar to conjecture-stating skills discussed earlier. Skills for finding a proof were more fully addressed in Koedinger and Anderson (1990), in which we presented a cognitive model and computer simulation of expertise in high-school geometry proofs.

In addition to needing skills for performing investigation and deduction in the service of argument, students need knowledge to make conclusions from these strategies, decide between them, and check their conclusions:



**Drawing conclusions:**

- If I find a counterexample, the conjecture is false.
- If I find a proof, the conjecture is true.

**Switching strategies:**

- If many attempts to find a counterexample fail, perhaps the conjecture is true, and I should try to prove it.
- If I can't find a proof, perhaps the conjecture is false, and I should look for a counterexample.

**Checking for errors:**

- If I've found a counterexample, I should still check that I cannot find a proof.
- If I've found a proof, I should still check extreme examples to make sure there is no counterexample.
- If I seem to have a proof and a counterexample, perhaps I've formulated the proof problem wrong or made incorrect measurements in my counterexample.

This higher-order knowledge about the nature of argument is lacking in many students.

The model of conjecturing proposed here is intended to fall somewhere between a descriptive model of existing student conjecturing skills and a normative model of what these conjecturing skills should be. The attempt is to characterize the edge of high-school student competence, discover which skills are present and which are lacking, and provide a direction for instructional design efforts. In the following section, we discuss some instructional implications of the model and, in particular, the role of modern software in learning conjecturing.

## MODEL-BASED DESIGN OF CONJECTURING ACTIVITIES AND SOFTWARE

This section provides suggestions for instructional activities and software directed at helping students acquire better conjecturing and argumentation skills. Most of the software suggestions concern ways to use existing tools, but I also propose the design of new tools to address aspects of conjecturing not well addressed by existing software.

### Activities and Software to Enhance Investigation

This section illustrates suggestions for software-related activities for Investigation using the Kite activity as an example. Such an activity could also be pursued with paper and pencil. I discuss how the activities differ and what advantages and disadvantages the technology brings.

*Model Construction and Measurement Activities.* I argued earlier that instruction in traditional compass and straight-edge construction is too isolated and overemphasized in typical high-school curricula. Other means of construction can get students more quickly involved with activities directed toward more general and powerful conjecturing and argumentation skills. The common placement of construction at the beginning of courses does not allow students to acquire the goal structures that frame construction techniques as useful knowledge. Instead students acquire construction skills within an arbitrary school-imposed goal structure and, as we saw in the Kite study, are not able to access these skills when they are truly needed. By starting with simpler approaches to construction, students can learn, first, when and where construction skills are needed. More specialized approaches to construction can emerge later in the curriculum with "pull" activities, in which there is a felt need for more efficient methods or more precise diagrams.

I use *diagram drawing* to refer to the simplest approach to diagrammatic modeling, in which figures are drawn by any method or tool set, and *diagram construction* to refer to the use of Euclidean methods and a restricted tool set. Examples of paper-based diagram drawing, from freehand to marked-ruler use, were given earlier. Students in the Kite study showed a substantial preference for diagram drawing over construction. When construction occurred, it was often prompted by other needs (e.g., to create a nonrhomboid kite).

*Computer-Based Diagrammatic Modeling.* The distinction between diagram drawing and diagram construction in computer-based tools is illustrated in Figs. 13.5 and 13.6. The kite in Fig. 13.5a was drawn, whereas the one in Fig. 13.6a was constructed. The steps to create Fig. 13.5a were (a) use the segment tool to draw four connected segments, (b) use the distance-measure tool to measure these segments, and (c) use the selection/move arrow to adjust the points until the segments AB and BC are equal, and AD and CD are equal. The steps to create Fig. 13.6a are analogous to a Euclidean construction: (a) draw a circle with center B that will determine the lengths of segments AB and BC, (b) draw a second circle with center D that intersects the first and determines the lengths of segments AD and CD, and (c) draw segments from the circle centers, B and D, to the two points where the circles intersect (points A and C).

Tools like Geometer's Sketchpad (Jackiw, 1991) make it easier to perform lower-level drawing, construction, and measurement steps. Thus, students can better focus on conjecturing and argumentation. Another advantage of these tools is that diagrams are "dynamic," making it easy for students to perform numerous geometry experiments. The diagrammatic models in Figs. 13.5 and 13.6 are dynamic in two ways: (a) the diagrams can be moved with their structural features maintained, and (b) the measurements of parts are dynamically updated. I illustrate two simple experiments with the drawn and constructed kites in Figs. 13.5 and 13.6.

Diagram drawing, besides being easier to do, has the advantage of facilitating the identification of "trivial conjectures" described earlier. Figure 13.5b shows the result of moving point A to modify the diagram in Fig. 13.5a. The diagram is no longer a kite, but one of the conjectured conclusions about kites, that the angles sum to 360°, still appears to be true. Thus, the experiment shows that this conclusion is not a characteristic particular to kites.



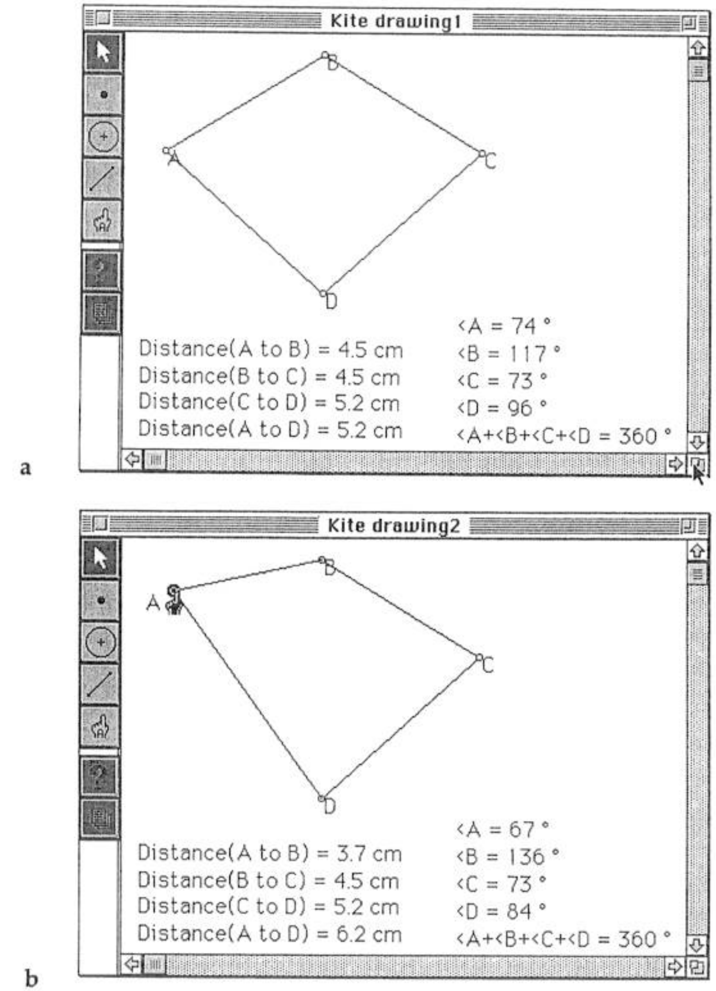
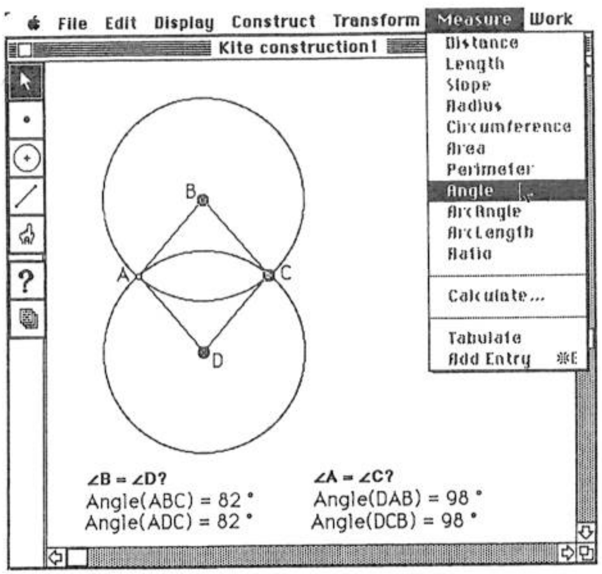



FIG. 13.5 Kite drawing (student work). (a) The points have been adjusted so that AB = BC and CD = AD. This example is consistent with the conjecture "The angles of a kite add to 360°." (b) Moving point A creates a figure that is no longer a kite. The angles still add to 360°, indicating that this property is not particular to kites.

The disadvantage of diagram drawing is that it is difficult to get the premises exactly right. (In some cases it is impossible; e.g., an equilateral a

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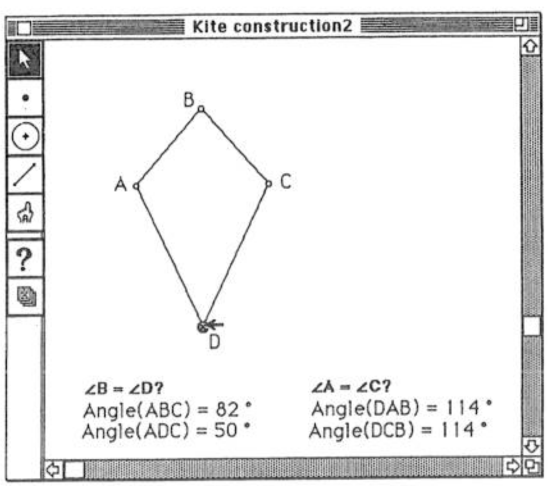



FIG. 13.6 The construction and measurement of a kite in Geometer's Sketchpad. (a) The Measure menu is illustrated, showing that when three points are selected (B, C, D in order) the angle formed by them can be measured. The result is recorded on the screen (e.g., "Angle(DCB) = 98°" at the bottom right). (b) The result of experimenting by moving point D. Sketchpad updates the measurements: Now angles B and D are no longer equal, but angles A and C still are. The student has hidden the circles used in the construction to better focus on the figure itself.



triangle cannot by drawn precisely on the pixel grid of a computer screen). A student must carefully move points and watch the measurements change until the conditions of the premises are met. Further, each new experiment usually requires that the same painstaking process is performed again. For instance, to get Fig. 13.5b back to being a kite, say a nonconvex one, would require moving points A and D until both AB = BC and AD = CD again.

Diagram construction like that in Fig. 13.6a facilitates experimentation because the diagram not only *looks* like a kite, but it *behaves* like one too. Sketchpad animates the movement of the diagram while maintaining the constraints of the construction: Although other things may change, segments AB and AC always stay equal because they are radii of the same circle—similarly for segments BD and CD. Sketchpad also dynamically updates the measurements as the diagram moves. Thus, in Fig. 13.6b we see the result of moving point D and the updated measurement values. Note that the student has "hidden" the circles used in the construction (see Fig. 13.6a). The figure still behaves in the same way, but this "hide" feature of Sketchpad allows the student to better focus on the figure itself. As a result of this experimentation, students may begin to modify their initial conjecture ideas and decide that although two of the opposite angles (∠A and ∠C) are congruent, the other two (∠B and ∠D) are not.

One can see from this example that Sketchpad simplifies the process of investigation in that a single construction can serve as a generator for multiple experiments. This kind of functionality is a general feature of many modern computational tools (e.g., spreadsheets, computer-aided design), in which setting up a system of constraints gives the user the power to automatically address "what if?" questions. Such dynamic models are a clear advantage over paper-based tools where exploring multiple models is considerably more time-consuming.

*A Prototype Intelligent Tutor Agent for Investigation.* We created a prototype cognitive Tutor Agent for diagram construction in Sketchpad (Ritter & Koedinger, 1995). Figure 13.7 illustrates the prototype system on the first exercise in chapter 3 of *Discovering Geometry:* copying a segment. The window at the top is Sketchpad, and the Messages window below is provided by the Tutor Agent.

Instruction in investigation, whether from teacher, text, peer, or computer tutor, needs to address two general classes of student errors: those involving the use of the tool, and those involving the mathematical content. The Tutor Agent helps with both types of errors. Providing automated remediation of low-level errors frees both students and teachers to focus on higher level content issues. Students may also get stuck in problem solving. As illustrated in Fig. 13.7, students can request a hint from the Tutor Agent, which can provide successively more specific hints, much like those in the Kite Interview Form, but specialized to each student's particular approach to a problem.

To summarize, computer-based tools for diagrammatic modeling such as Geometer's Sketchpad offer some increased benefits and reduced costs over paper-based tools. The key benefit is that computer diagram-

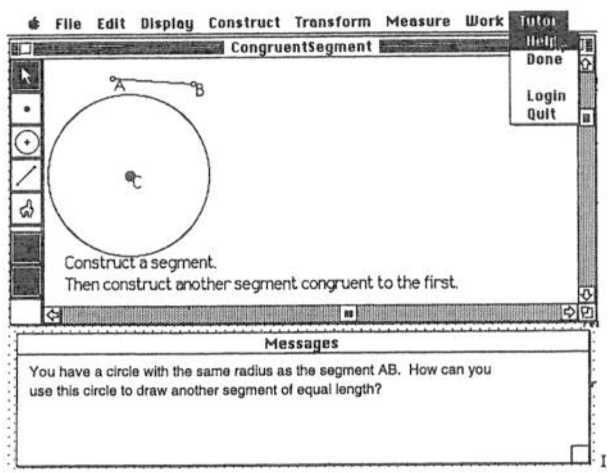



FIG. 13.7 Geometer's Sketchpad and an intelligent Tutor Agent for geometric construction. The rightmost Tutor menu was added to Sketchpad and allows the student to request hints from the Tutor Agent. Hints then appear in the Messages window.

ming allows for the creation of a dynamic model through which investigation is greatly facilitated by making it easy for students to modify the diagrammatic model. On the cost side, many low-level concerns are reduced or eliminated by providing automated drawing, measurement, and construction capabilities. Errors due to compass slippage or protractor misalignment are eliminated. Computer diagramming does have its own unique tool errors, though. Such difficulties can be intimidating to students and often even more so to teachers, who may encounter conflicts with their paper-based construction methods. The creation of intelligent Tutor Agents can help reduce these barriers for both students and teachers.

## Activities and Software to Enhance Conjecturing

Although software tools for the other skill components are available, there is nothing that directly supports conjecturing. What is needed is a "Conjecture Editor" tool that aids students in formulating conjectures and recording them for future reference. By analogy to structured editors in computer programming environments, a Conjecture Editor could provide support through displaying a list of context-sensitive conjecture templates. The need to support students in conjecture formulation is evidenced by the approach in the *Discovering Geometry* textbook, in which students are given nearly completed conjectures to complete. For example, in the first discovery, which investigates the properties of the angles formed by crossing lines, students are prompted to make a conjecture that starts: "If two angles are vertical angles, then the angles are____." Teachers whom I have worked with at the urban schools report that when students are not given such prompts, they have great difficulty in expressing their conjectures. However, such prompts have negative cognitive consequences. Students are not engaged in the target skill of conjecturing, but in a fill-in-the-blank task. Further, students lose the experience of creation and discovery that conjecturing activities are intended to achieve. A Conjecture Editor could preserve the desired role of student as creator yet provide subtle support to make the task within reach of students.

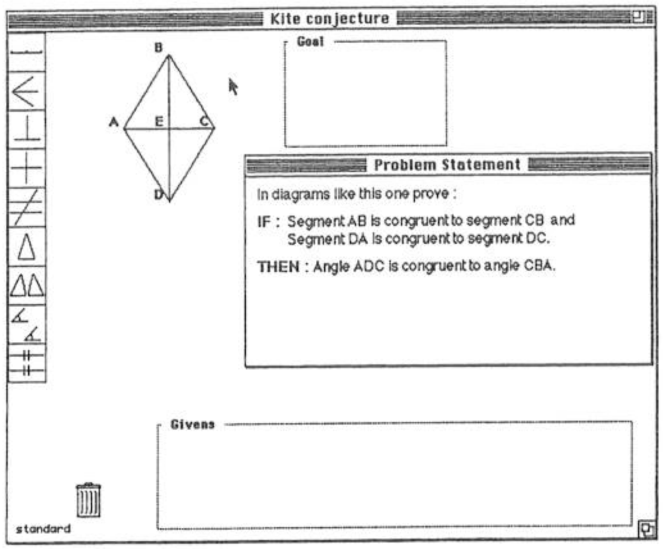


## Activities and Software to Enhance Argumentation

Following students' greater readiness for empirical argument over deductive proof, initial activities should emphasize empirical argument. Such arguments are a natural extension of conjecturing activities, but students need to be pressed to provide multiple examples in support of their conjectures. Further, the importance of argument processes will not be recognized by students unless they have numerous experiences with conjectures that seem true but are false. We have used what we call "Truth Judgment" activities to engage students in argument processes by asking them to evaluate conjectures that may or may not be true: for instance, "Is the following conjecture always true: 'The diagonals of a quadrilateral cross.' " The conjecture-evaluation skills needed for Truth Judgment problems are skills scientists use in reviewing a paper or listening to a talk. Related skills are used in evaluating claims in a newspaper article or a political speech.

Both the Investigate and Deduce strategies are relevant to Truth Judgment problems, but students are more likely to engage initially in investigation. A typical student or class might begin by arguing the conjecture just given is true by drawing a diagrammatic model of a quadrilateral, likely a rectangle, with crossing diagonals. When pressed "Do the diagonals always cross?," students will draw multiple quadrilaterals. Because of a bias toward convex quadrilaterals, it is likely that many students will become convinced the conjecture is true. Teachers need to wait and fight their urge to help, giving only vague hints if necessary (e.g., "Have you tried all kinds of examples, including weird ones?"). The more time it takes for a student to find a counterexample and convince others, the deeper will be the lesson. Activities like this, where intuitions are misleading, enable students to experience the utility of the argument process. In traditional geometry classes, students get few such opportunities because the predominant activity is to prove conjectures that are already known to be true.

Earlier we illustrated the use of Geometer's Sketchpad to support the Investigation process to help both generate and test conjectures. In previous work we built a software learning environment (ANGLE), which provides a graphical interface for discovering geometry proofs and online intelligent help (Koedinger & Anderson, 1993a, 1993b). Here we illustrate how ANGLE supports the Deduce process, not only for the usual purpose of verifying a conjecture, but also in finding evidence against a conjecture and in facilitating the discovery of a new conjecture. Figure 13.8a shows

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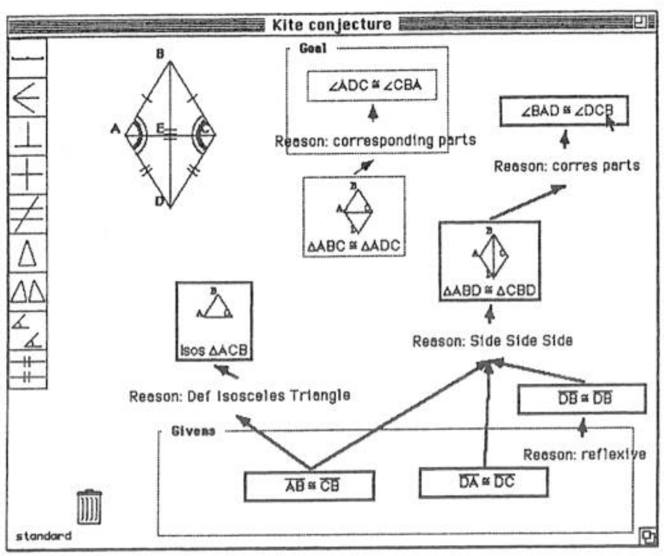


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FIG. 13.8 Student work in ANGLE, an editor and intelligent tutor for geometry proof. (a) The student has entered a conjecture. (b) The student has been unable to find a proof for the entered conjecture ($\angle ADC \cong \angle CBA$), but the attempt leads to a discovery (and proof) of a related conjecture ($\angle BAD \cong \angle DCB$).



the ANGLE screen with a student-entered conjecture for the Kite problem—the most frequent false conjecture from the Kite study. Figure 13.8b shows the screen some time later. After attempts to work forward from the given (e.g., proving triangle ACB is isosceles) and backward from the goal (e.g., posing the subgoal to prove triangles ABC and ADC congruent), the student can find no way to prove her conjecture. In the process, however, she deduces a new conjecture by finding she can prove triangles ABD and CBD congruent and, in turn, angles BAD and DCB congruent.

Students can learn from independent use of ANGLE, but ANGLE is more effective when well integrated into a classroom curriculum. In a laboratory study where students worked independently with ANGLE for a total of 8 hr, we found their proof-writing achievement increased dramatically (by 60%) from pretest to posttest (Koedinger & Anderson, 1993a). In a classroom study, students using ANGLE and Truth Judgment problems with an experienced teacher performed one standard deviation better than students in classes with the same teacher but without ANGLE (Koedinger & Anderson, 1993b).

The Conjecture Editor, described earlier, could be combined with ANGLE to create a "Conjecture Manager" tool. This tool not only would provide students with the capabilities to enter conjectures and proofs, but also would provide a way of recording new discoveries and building on old ones. As students progress through the curriculum, they should essentially "build a book" (see Healy, 1993) by recording conjectures, labeling some conjectures as postulates, and proving other conjectures using these postulates. Figure 13.9 shows a mock-up menu for the Conjecture Manager. The Conjecture menu would allow students to "enter" new conjectures, "see" previous conjectures, promote some as "postulates," and "prove" others as theorems.

Choosing the Prove menu item and selecting a conjecture would initiate ANGLE, in which students would draw a reference diagram, enter the conjecture with reference to this diagram as Givens and a Goal, and then search for a proof. The National Council of Teachers of Mathematics (NCTM) Standards emphasize developing "short sequences of proofs" that show how one conjecture can support another (NCTM, 1989). A Conjecture Manager would facilitate a build-a-book approach (Healy, 1993) in which students could create their own sequencing of definitions, postulates, and theorems in geometry.

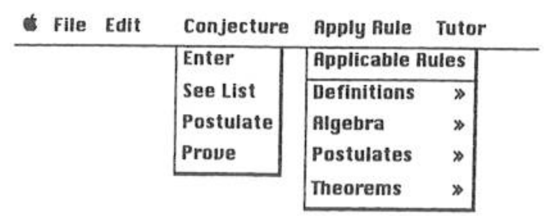


FIG. 13.9 Menu items in the proposed Conjecture Manager.

## CONCLUSION

I have presented a model of conjecturing and argumentation skills based on the analysis of student problem solving in an open-ended conjecturing task. Conjecturing is the process of generating generalizations about a class of phenomena, whereas argumentation is the process of finding support for a generalization. Both processes are supported by two complementary reasoning strategies: model-based investigation, and rule-based deduction. Conjectures can be generated by investigating models of the phenomena and inducing new conjectures, or by chaining together prior "rules" (well-supported conjectures) and deducing new conjectures. Similarly, arguments for a conjecture can cite inductive support in positive examples (models) of the phenomenon, or deductive support in the chaining together of prior rules that show the logical link between the premise and conclusion of the conjecture.

This model of conjecturing and argumentation has similarities to cognitive models of scientific discovery (e.g., Klahr & Dunbar, 1988; Langley, Simon, Bradshaw, & Zytkow, 1987) and argumentation (e.g., Cavalli-Sforza, Lesgold, & Weiner, 1992). It is distinguished from these previous models in its integration of both empirical and deductive methods for discovery and argumentation.

In this chapter, I have attempted to provide evidence that the ability to discover new ideas and develop convincing arguments is not talent, but the consequence of particular skills and knowledge. The importance of such skills goes beyond mathematical discovery; conjecturing and argumentation skills are also relevant for nonroutine problem solving and for aiding recall. The model-and-measure strategy (see Fig. 13.4) illustrates the application of conjecturing skills in problem solving. Recall can be aided by both conjecturing and argumentation skills because forgotten facts can be rediscovered or rederived.

Not only are conjecturing and argumentation skills relevant for tasks besides discovery, they are also relevant in other domains. Further, there is evidence that such skills are lacking in the adult population. Many of the reasoning difficulties we observed in the Kite study show up in the reasoning patterns of adults arguing for claims outside of mathematics. Kuhn (1991) studied the argumentation skills of 160 U.S. adults who were asked to defend claims about familiar social policy issues (e.g., causes of school failure or unemployment). She found that more than half have poor reasoning abilities; in particular, they fail to make arguments that rely on evidence or accepted generalizations in a sound way. Like many students in the Kite study, adults in Kuhn's study tended to cite single examples (often personal experiences) as sufficient support for their claims. Both groups showed signs of confusion about the difference between claims and evidence. In the Kite study, many students had difficulty seeing conjectures as more than statements about a particular diagram, that is, as general claims about all kites. In the Kuhn study, many adults talked about claims and examples as one and the same (e.g., I believe school failure results from poor teaching because when I was in school I had a lot of bad teachers). Kuhn (1991) commented that for many people "the evidence is not sufficiently differentiated from the theory itself" (p. 285). She suggested that "to progress beyond the fusion of theory and evidence to the full differentiation and coordination of the two requires . . . thinking about theories, rather than with them, and thinking about evidence and its bearing on a theory, rather than merely being influenced by it" (Kuhn, 1991, p. 285).



If students who pass geometry tend to acquire better abilities to differentiate and coordinate claims and evidence, then Pelavin (1990) may be correct in his claim that high-school geometry provides a unique opportunity to learn fundamental reasoning skills that are important for college success. This claim is consistent, though not uniquely so, with the results of his College Board study (Pelavin, 1990) where he found that among a number of other potential factors, passing high-school geometry is the best correlate with college success. An alternative explanation for this correlation is that high-school geometry courses are uniquely difficult and thus have served as a selection filter through which flow only those students who enter the course with the reasoning and learning skills needed for success in this difficult course and in college. Between these extremes is the likely reality. Some general reasoning and learning skills are probably improved through successful experiences in geometry. We need further empirical and theoretical research (cognitive models) to identify clearly what these skills are and the extent to which they can be acquired generally (i.e., not tied to the domain of instruction). Attempts to teach general reasoning skills have often failed, but successes at "transfer" have had in common a clear identification of the targeted general skills, often in the form of a cognitive model, and this model has been the basis for careful instructional design (e.g., Klahr & Carver, 1988; Lehrer, Randle, & Sancilio, 1989; Schoenfeld, 1985; Singley & Anderson, 1989).

One example of a class of skills that appears relevant both in geometry and more generally is skills for "set breaking" that can be used to avoid rigid thinking and lead to more flexible problem solving (Luchins & Luchins, 1959). Geometry activities can help students address their own biases toward prototypical figures and "perceptual set." How well general set-breaking skills and heuristics can be acquired in geometry is not clear, but it is clear that such skills cannot be acquired without opportunities that illustrate the need for them. The Kite problem is one such activity. It provides a good context for instruction on heuristics like the "general model heuristic": If your goal is to construct and investigate an example or model of a phenomenon (e.g., a kite), then construct as general an example as possible—avoid adding properties or constraints that are not required (e.g., making all four sides equal).

Designing instruction to help students learn discovery skills should be distinguished from a discovery learning approach. Just giving students discovery tasks and little support can lead to interesting results, but is too slow and frustrating for many students (Healy, 1993). Instead, students should learn discovery skills "by doing" in a supportive environment deliberately designed to achieve the skills outlined in the cognitive model. The elements of this environment should include well-chosen activities that elicit student thinking and classroom debate, well-timed instructorfacilitated introduction of better conjecturing and argumentation techniques, and the use of modern software tools that facilitate the conjecturing process and that include, where possible, the additional just-in-time support of a computer-based Tutor Agent.

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