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An Investigation of Teachers' Beliefs of Students' Algebra Development

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Elementary, middle, and high school mathematics teachers (N = 105) ranked a set of mathematics problems based on expectations of their relative problem-solving difficulty. Teachers also rated their levels of agreement to a variety of reform-based statements on teaching and learning mathematics. Analyses suggest that teachers hold a *symbol-precedence* view of student mathematical development, wherein arithmetic reasoning strictly precedes algebraic reasoning, and symbolic problem-solving develops prior to verbal reasoning. High school teachers were most likely to hold the symbol-precedence view and made the poorest predictions of students' performances, whereas middle school teachers' predictions were most accurate. The discord between teachers' reform-based beliefs and their instructional decisions appears to be influenced by textbook organization, which institutionalizes the symbol-precedence view. Because of their extensive content training, high school teachers may be particularly susceptible to an *expert blindspot,* whereby they overestimate the accessibility of symbol-based representations and procedures for students' learning introductory algebra.

The study of people engaged in cognitively demanding tasks must consider the relation between people's judgments and actions and the beliefs they hold. Several aspects of people's decision making are well established. People do not strictly follow the laws of logic and probability when weighing information or following im-

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plications (Cheng, Holyoak, Nisbett, & Oliver, 1986; Wason & Johnson-Laird, 1972). In fact, most of the time, human decision making differs substantially from the normative logical process (Kahneman & Tversky, 1973; Rosch, 1973; H. A. Simon, 1969; Tversky & Kahneman, 1973). These characteristics, coupled with an appreciation of the inherent limitations of human attention, short-term memory, and cognitive processing (e.g., Baddeley & Hitch, 1974; Just & Carpenter, 1996; Miller, 1956) have led researchers of complex cognitive behavior to regard human decision making as "reasonable," rather than rational (Borko & Shavelson, 1990).

One area of complex cognitive behavior that is of particular interest is the domain of teaching. Many studies have focused on identifying links between teachers' decisions and actions and the knowledge and beliefs that are hypothesized to mediate them. Investigators have found that teachers' interpretations and implementations of mathematics curricula, for example, are greatly influenced by their knowledge and beliefs about instruction (Ball, 1988; Borko et al., 1992; Clark & Peterson, 1986; Raymond, 1997; Romberg & Carpenter, 1986; Thompson, 1984), mathematics (Cooney, 1985; Raymond, 1997), and student learning (Ball, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, Carpenter, Franke, & Carey, 1992; Romberg & Carpenter, 1986).

Because the beliefs that teachers hold are so instrumental in shaping mathematics teachers' decisions and actions, it is important that these beliefs be a focus of educational research; in addition, because these decisions affect students' learning experiences so directly, it is equally important to understand the accuracy of teachers' beliefs. In this study we examine the congruence of teachers' beliefs about mathematical problem solving with respect to students' actual problem-solving performance, and investigate possible influences that can affect teachers' instructional decisions.

Only a few studies have looked specifically at the relation between teachers' beliefs about student reasoning and students' actual problem-solving behaviors and levels of performance (e.g., Carpenter, Fennema, Peterson, & Carey, 1988; Peterson, Carpenter, & Fennema, 1989; Peterson, Fennema, Carpenter, & Loef, 1989; Wigfield, Galper, Denton, & Seefeldt, 1999). For example, members of the cognitively guided instruction project (Carpenter et al., 1988) found that first-grade teachers' beliefs about student problem solving were usually consistent with students' performance and with the general strategies that students used. However, teachers underestimated the frequency with which students used counting strategies and overestimated their use of derived facts and direct modeling methods. Despite these inaccuracies, teachers' beliefs of student problem-solving success were significantly correlated to their students' actual performance. Peterson, Fennema et al. (1989) showed that teachers' beliefs about student performance were most accurate when the beliefs were organized within a "cognitively-based perspective." This meant they believed that children construct their own mathematical knowledge, that skills are best taught within problem-solving contexts, and that instruction should be developmentally informed and organized to facilitate students' construction of knowledge.

Much of this research has focused on elementary-level mathematics and has greatly enhanced our understanding of elementary school teachers' beliefs. Lacking, however, is a similar emphasis at the secondary level.[1] One study (Nathan and Koedinger, 2000) analyzed the expectations high school mathematics teachers had for algebra students' problem-solving performance. Nathan and Koedinger found that teachers accurately judged students' performance abilities on some types of problems but systematically misjudged them on others. They asked teachers to rank order the relative difficulty of six types of mathematics problems that varied along two dimensions. Table 1 shows a sample problem of each type, organized by the two major factors. Along the first dimension, the unknown value was either the result of a problem (P4, P5, and P6) or positioned at the beginning of the problem (P1, P2, and P3) so that it was expressed in relation to other known quantities. The second dimension included problems in one of three presentation formats: verbal with a context (a story problem; P1 and P4), verbal with no context (a word equation problem; P2 and P5), or symbolic (a symbol equation; P3 and P6).

Task analyses of problems along the position-unknown dimension show that result-unknown problems are solvable by direct application of the arithmetic operators or by modeling, whereas start-unknown problems defy arithmetic forms of modeling and are typically considered to be algebra-level problems (Tabachneck, Koedinger, & Nathan, 1995). Correspondingly, children and adults typically have much lower levels of performance with start-unknown than with result-unknown problems (Carpenter & Moser, 1983; De Corte, Greer, & Verschaffel, 1996; Fuson, 1988; Heffernan & Koedinger, 1997; Koedinger & Nathan, 1999; Riley, Greeno, & Heller, 1983).

Koedinger and Nathan (1999) analyzed the problem-solving performance of high school students ($N = 76$) who had completed a year of algebra. The problems were like those presented in Table 1. In accord with prior research, Koedinger and Nathan found that the students' level of performance on start-unknown (i.e., algebra) problems was significantly below that of result-unknown (i.e., arithmetic) problems (see Table 2, Column 6). This general finding was replicated with a new group of algebra students the following year ($N = 171$). This pattern of perfor-

[1]Recent international comparisons of academic performance indicate the need to improve our understanding of instructional practices at the secondary level and across the span of Kindergarten through Grade 12 education. For example, the Third International Mathematics and Science Study (National Center for Educational Statistics, 1996) showed that the level of performance of U.S. students is above the international average at 4th grade (out of 26 nations), slightly below average by 8th grade (41 nations), and near the bottom of the distribution by 12th grade (out of 21 nations). It appears from these data that, in relation to other nations, the effectiveness of mathematics education in the United States decreases with increasing grade level.

TABLE 1
The Problems Given to Students and Teachers Can Be Organized by the Presentation Type (Three Levels) and the Position of the Unknown Value (Two Levels)

	Presentation Type		
	Verbal Problems		
Unknown Value	*Story*	*Word*	*Symbol*
Result-unknown (arithmetic)	P4: When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hr he worked and found his hourly wage. How much per hour does Ted make?	P5: Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?	P6: Solve for x: $(81.90 – 66) / 6 = x$
Start-unknown (algebra)	P1: When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hr he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour does Ted make?	P2: Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?	P3: Solve for x: $x \times 6 + 66 = 81.90$

Note. P = problem.

mance was accurately predicted by the majority (84%) of high school teachers (n = 67) in the sample (Nathan & Koedinger, 2000).

The high school students in Koedinger and Nathan's (1999) studies also showed a pronounced performance advantage for solving verbally presented problems (story problems and word equations) over symbolic equations, $F(2, 74) = 12.6$, $p < .0001$ (see Table 2). This finding has been replicated with students in several populations across sixth through ninth grades in several regions of the United States, as well as with adults (Koedinger & Nathan, 1999; Tabachneck, Koedinger, & Nathan, 1995; Verzoni & Koedinger, 1997). Although the performance difference was large, teachers' predictions along this dimension were generally inaccurate. The majority of high school teachers (70%) ranked verbally presented problems as more difficult than symbol equations for both arithmetic and algebra (Nathan & Koedinger, 2000). These data suggest that high school mathematics teachers hold beliefs that cause them to systematically misjudge students' symbolic- and verbal-reasoning abilities. These beliefs could lead teachers to make poor decisions regarding curriculum, instruction, and assessment. Thus, it is important to consider the influences that may shape teachers' views.

TABLE 2
Summary of Teachers' Average Rank Ordering of Problem Types and Students' Performances

Rank	*All Teachers*[a]	*Elementary Teachers*[b]	*Middle School Teachers*[c]	*High School Teachers*[d]	*Student Performance*[e]	*% Correct*[e]
Easiest[f]	Arithmetic–Word–Equation (P5)	Arithmetic–Word–Equation (P5) Arithmetic–Equation (P6)	Arithmetic–Word–Equation (P5)	Arithmetic–Word–Equation (P5) Arithmetic–Equation (P6)	Arithmetic–Story (P4) Arithmetic–Word (P5)	73% 67%
Middle[f]	Arithmetic–Story (P4) Arithmetic–Equation (P6)	Arithmetic–Story (P4) Algebra–Equation (P3)	Arithmetic–Story (P4) Algebra–Story (P1)	Algebra–Equation (P3)	Algebra–Story (P1) Algebra–Word (P2) Arithmetic–Equation (P6)	59% 54% 53%
Hardest[f]	Algebra–Story (P1) Algebra–Equation (P3) Algebra–Word–Equation (P2)	Algebra–Story (P1) Algebra–Word–Equation (P2)	Algebra–Word–Equation (P2) Algebra–Equation (P3) Arithmetic–Equation (P6)	Arithmetic–Story (P4) Algebra–Word–Equation (P2) Algebra–Story (P1)	Algebra–Equation (P3)	37%

Note. P = problem.
[a]$n = 105$. [b]$n = 36$. [c]$n = 30$. [d]$n = 39$. [e]$n = 76$. Student data are from Koedinger and Nathan (1999). [f]Difficulty divisions indicate significant differences in teacher ranking or student performance, $p < .05$.

INFLUENCES ON TEACHERS' BELIEFS

Some researchers have speculated on the conceptual bases for teachers' beliefs. Peterson, Fennema, et al. (1989) attributed incongruencies between teachers' beliefs and students' performance to poor pedagogical content knowledge, whereas Borko et al. (1992) additionally pointed toward insufficient content knowledge on the part of the teacher. We consider two additional influences that deserve consideration: the curricular and professional standards for mathematics instruction and the structure and content of school textbooks.

Principles of Mathematics Educational Reform

In this era of mathematics education reform, the curricular and professional standards of the National Council of Teachers of Mathematics (NCTM, 1989, 1991) serve as a foundation for the newly stated principles of mathematics learning, content, and teaching. To support students' learning of mathematics, the curriculum standards call for an approach that instills an appreciation of the role of mathematics in society and builds students' confidence for reasoning with mathematics and applying mathematical ideas and methods. In the *Curriculum and Evaluation Standards*, mathematics is presented as a multifaceted tool for solving problems and reasoning formally, as well as a medium for communication (NCTM, 1989). Students should develop a conceptual understanding of numbers, symbols, and procedures that is robust enough to promote mathematical and scientific learning and reasoning in novel settings. For algebra, this translates to curricula that encourage reasoning about unknown quantities and generalized relations; modeling situations and abstract relations with symbols and solution methods that are imbued with meaning; and working with a variety of representational forms, including equations, tables, diagrams, and verbal relations.

To support these curricular goals, teachers are encouraged to see students as active constructors of knowledge, which students build up through exploration, invention, and discourse with other members of their mathematical learning communities, as well as through interactions with various manipulatives and forms of technology. According to the *Professional Standards* (NCTM, 1991), as active learners, students should be invited to make connections between different methods and representations and should be able to construct methods for novel problems from their prior understanding. Students should not come away believing that there is only one way to represent or solve all problems in mathematics. The emphasis on students' reasoning and problem solving over memorization also means that teachers should see students' methods as paramount to their thinking, and well-articulated logical approaches should overshadow the correctness of an answer that cannot be clearly justified.

Much about these ideas can be summarized in a few principles: (a) Teachers should focus on students' solution process more than the product of their calculations, (b) there are many ways to approach a given problem, (c) invented solution methods can be effective, (d) teachers should encourage students to invent methods and perform other activities that foster knowledge construction, and (e) the use of alternative solution approaches is indicative of students' adaptive methods of reasoning. In our study, we examined teachers' levels of agreement with principles like these as we documented teachers' general beliefs about learning and instruction and the relation of these beliefs to views of student problem solving.

The Influence of Textbooks

Mathematics textbooks have been suggested as another source of influence on teachers' beliefs. Nathan and Koedinger (2000) proposed the symbol-precedence hypothesis as the basis for high school teachers' assessments of early algebra problem difficulty. This hypothesis suggests that teachers operate with a view of mathematical development that assumes that students' symbolic-reasoning developmentally precedes verbal-reasoning ability. Within the symbol-precedence view, symbolic reasoning is seen as a form of "pure mathematics" and is considered as a necessary prerequisite for more advanced verbal "applications." Although this view is considered to be contrary to the principles of mathematics curricular and instructional reform discussed earlier, it is implicit in many widely used pre-algebra and algebra textbooks, which tend to introduce new material in a symbolic format and then later assign word equations and story problems as challenge activities (for details of this investigation, see Nathan & Long, 1999).

Nathan and Koedinger (2000) provided some preliminary evidence for the symbol-precedence hypothesis using Kendall's rank correlation test. The test revealed a significant level of agreement between teachers' views and the organization of algebra textbooks, $\tau(12) = .867$, $p < .02$. This indicated that high school teachers' judgments of symbolic and story problem difficulty mirrored the symbol-precedence sequencing of problems shown in textbooks. One interpretation of this pattern is that, through reliance on and repeated exposure to mathematics textbooks, high school teachers internalize the symbol-precedence view that is institutionalized by textbooks and use it as a basis for predicting problem-solving difficulty for students.

Although textbooks tend to emphasize a symbolic-precedence approach for teaching algebra-level problem solving, analyses of students' solution strategies and errors have revealed widespread use of informal solution methods such as guess-and-test and unwinding (for more details of students' strategies, see Koedinger & Nathan, 1999; Nathan & Koedinger, 2000). These informal strategies may be considered proto-algebraic methods in that they contain some of the essential conceptual underpinnings of the formal symbol-manipulation methods. For example, the guess-and-test method models the problem situation and uses an iterative approach to substitute a chosen value for the unknown (the guess) until a value is reached that satisfies the quantitative constraints of the problem (e.g., Hall, Kibler, Wenger, & Truxaw, 1989). It instantiates the algebraic concept of a variable and shows the value of considering equations as mathematical structures. The unwinding method belongs to the class of working backwards strategies (e.g., Kieran, 1988; Polya, 1957). It strips away the given relations in the opposite order they are given, while at the same time inverting the mathematical operations. Unwinding essentially takes a multistep, start-unknown problem and turns it into a series of one-step, arithmetic calculations. In so doing, it provides a natural way of assigning meaning to quantities and intermediate results and embodies the notions of inverting operations.

The analyses of students' problem-solving processes performed by Nathan and Koedinger (2000) showed that verbal problems were easier for students to solve than symbolic problems because verbal problems were more likely to elicit informal strategies such as unwind and guess-and-test. These informal methods were executed more successfully than the formally taught symbolic procedures—even though all of the high school students in the original study had gone through a traditional Algebra I curriculum. Although formal symbol manipulation methods led to a correct answer on start-unknown problems 54% of the time, the guess-and-test strategy was correct 68% of the time, and the unwinding method was correct 72% of the time (Nathan & Koedinger, 2000). These analyses suggest that students enter pre-algebra instruction with quantitative reasoning abilities that are highly effective at solving algebra-level problems when they are presented verbally. Unfortunately, it also appears that high school mathematics teachers are largely unaware of the range and efficacy of students' informal solution strategies and tend to overestimate students' proficiency with formally taught symbolic-solution methods.

Because of the emphasis on verbal reasoning shown in students' invented solution methods and their preference to apply these methods to verbal problems, Nathan and Koedinger (2000) described students' developing algebra competence within a verbal-precedence model and contrasted it with the symbol-precedence model. In a quantitative comparison, Nathan and Koedinger showed that the verbal-precedence model gave a better account of students' early algebra performance than the symbol-precedence view exhibited by most teachers. The verbal-precedence model predicted the performances of 91% of the students (n = 76) on problems like those in Table 1 (and the performance of 88% of the students in the replication study, n = 171), whereas the symbol-precedence model predicted the performances of only 62% of students (46% of the students in the replication study).

Although Nathan and Koedinger (2000) offered a plausible account for why high school teachers' views of problem difficulty were at odds with student performance, they did not directly investigate whether teachers' beliefs in fact conformed to the symbol-precedence view implicit in textbooks, whether they aligned themselves with the principles of mathematics reform, and whether teachers' levels of agreement with any of these views correlated with their predictions about problem difficulty. This study was designed to address these issues and to extend our understanding of the nature of teachers' beliefs, their influences, and their variability across teacher grade levels.

Hypotheses of This Study

This study examines the accuracy of teachers' beliefs about algebra problem-solving difficulty and investigates how general beliefs about mathematics teaching and student learning influence teachers' judgments. The ranking task used by Nathan and Koedinger (2000) was employed, and data on the relative difficulty ranking of problems using the two dimensions of mathematical factors presented in Table 1 were collected from a new sample of teachers. Teachers' views about several reform-based issues of mathematical performance, learning, and instruction were also examined using a belief instrument that included a construct directly examining teachers' alignment with the symbol-precedence view of algebra development as typically presented in textbooks. To examine the generality of these findings, responses from participating teachers from a range of grade levels (elementary, middle, and high school) were studied.

We set out to test three hypotheses. First, drawing on previous research findings, we hypothesized that the high school teachers in our sample would make judgments about problem difficulty in accordance with the symbol-precedence view of mathematical development. Thus, we predicted that high school teachers will tend to rank order verbal problems as relatively more difficult to solve than matched symbolic-equation problems, and algebra problems as more difficult than arithmetic problems. Next, we address the extent to which the symbol-precedence view corresponds to the decisions of teachers of different grade levels. Because the symbol-precedence view seems to be so influential, we expected to find that the judgments of elementary and middle school teachers will be similar to those made by high school teachers. Finally, we examine the relation between teachers' general views of learning and instruction and teachers' task-specific judgments concerning algebra-level problem difficulty. We hypothesize that teachers' judgments about algebra problem-solving difficulty will be correlated with their levels of agreement with reform-based statements on pedagogy, learning, problem solving, and mathematical development.

METHOD

Participants

Participants of this experiment were 107 Kindergarten through Grade 12 teachers from a single district who attended an obligatory school district-sponsored workshop for either primary or secondary mathematics teaching during the fall. Of the original 107 participants, 2 produced forms that were insufficiently completed (less than half of the difficulty rank or belief statements had responses); therefore, those data were removed from the sample prior to analysis. This left a final count of 105 participants. All participants were either mathematics teachers (middle and high school grades) or elementary school teachers who taught all of the major content areas, including mathematics. The teachers' reported grade levels of instruction ranged from 2nd to 12th. The students in this district were predominantly White (20% have minority status) and lived in predominantly suburban and middle-class urban areas.

Design, Materials, and Procedure

Teachers received a sheet with the difficulty ranking activity (see the Appendix). The teachers were asked to rank order six mathematics problems, from easiest to most difficult. The specific problems given to the participants are shown in the Appendix (see Table 1 for the underlying problem structure). These problems were chosen because they were representative of problems found in pre-algebra and algebra mathematics textbooks. Teachers were asked to "rank order the 6 problems shown starting with the ones you think are easiest for your students, and ending with the ones you think are hardest." Ties were allowed.

In addition to the ranking task, the teachers in this sample were given a set of 47 statements. They were given 20 min to rate the degree to which they disagreed or agreed with each statement by selecting the appropriate number along a 6-point Likert scale. Larger numbers indicated greater disagreement. Participants received the 47 items in a randomized order and were asked to "Circle the number to the right that corresponds most accurately with your beliefs about the accompanying statement." The 6-item scale ranging from 1 (*strongly agree*), 2 (*agree*), 3 (*agree more than disagree*), 4 (*disagree more than agree*), 5 (*disagree*), to 6 (*strongly disagree*) was presented at the left margin of each statement.

The 47 items formed six constructs (item groups). Examples of each construct are presented in Table 3. Wherever possible, each construct included items that were worded both positively (affirming the construct) or negatively (negating the construct). Many of the items were adapted from previously published work, including Witherspoon and Shelton (1991), Cobb (1990), and Peterson, Fennema, et

TABLE 3
Summaries and Example Items From the Six Different Survey Constructs Used for the Survey of Teachers' Views of Mathematics, Math Instruction, and Student Learning, Along With Sample Items Presented Positively and Negatively

Summaries	*Positive Item*	*Negative Item*
Algebra is best (11 items) presents the view that algebraic procedures are the singularly most effective method for mathematical problem solving.	Using algebra for story problem solving is the most effective approach there is.	There are many effective approaches to solving any algebra story problem, and manipulating symbols is only one method.
Invented solution methods are effective (7 items) presents the view that students can learn and invent effective methods for problem solving that may differ from those taught.	Students enter the algebra classroom with intuitive methods for solving algebra story problems.	Most students cannot figure out for themselves how to solve algebra story problems.
Symbol precedence view (6 items) holds the view commonly expressed in math textbooks that arithmetic problems are easier and need to be presented before algebra. Also, within a mathematics topic, math problems presented in words are most difficult and need to appear later in the curriculum.	Arithmetic story problems are easier for students to solve than algebra story problems.	Solving math problems presented in words should be taught only after students master solving the same problems presented as equations.
Teachers should encourage invented solution methods (8 items) states that students may possess valid ways of reasoning as they enter the classroom, and may figure out for themselves effective problem-solving approaches.	Students should be encouraged to invent their own methods to solve mathematics problems.	Rewarding right answers and correcting wrong answers is an important part of teaching.
Product over process (4 items) emphasizes correct answers over a student's reasoning process.	Getting the correct answer is a better indicator of learning than is the ability to articulate a good solution approach.	Mathematical understanding is more clearly shown in a student's reasoning than in the final answer a student produces.
Alternative solution methods indicate knowledge gaps (6 items) states that alternative (unschooled) methods such as arithmetic, guess-and-test, and other nonsymbolic methods demonstrate gaps in the student's knowledge.	When a student uses an arithmetic approach to solve an algebra word problem, that indicates a weakness in that student's math abilities. (positive)	Use of a "guess and check" strategy to solve an algebra story problem shows an adaptive approach to problem solving.

al. (1989). These items were chosen because they broadly addressed current reform-based issues of pedagogical practice, mathematical learning and development, problem solving, and the role of algebra in the domain of mathematics. Included were statements on the symbol-precedence view of algebra instruction discussed earlier. The constructs *invented solution methods are effective* and *teachers should encourage invented solution methods* provided the strongest declarations of student-centered and student-directed learning when stated in the affirmative. The remaining constructs, algebra is best, symbol precedence, product over process, and alternative solution methods indicate knowledge gaps, voiced views that were procedure- or curriculum-centered when stated in the affirmative.

Earlier pilot testing helped to establish unambiguous wording and construct formation.

Participants were told that the intent of the questionnaire was to learn about the views that teachers held regarding mathematics, learning, and instruction; that all information they provided would be kept confidential; and that only the collective results of the group would be shared. The belief instrument was administered immediately after completing the problem difficulty ranking task.

RESULTS AND CONCLUSIONS

Belief Instruments

Descriptive statistics and reliability analyses of the belief constructs were compiled. Five of the original 47 items were dropped as a result of the reliability analysis. Results are presented on the remaining 42 items grouped within the six original constructs.

The analyses of teachers' ratings show that reliability measures for the given items were generally high (Cronbach's alpha had a range of .65 to .84), indicating good agreement on items that were theoretically clustered together (see Table 4). The mean rank of the items (maximum = 6.0) tended to cluster around the middle range of the scale (3.5), indicating that the 6-point scale given was generally sufficient for the teachers to express their level of agreement adequately. (Note that higher mean scores indicate greater disagreement.) The one exception to this is the product over process construct (M = 5.1 out of 6.0 total points), which was skewed toward disagreement. This suggests that, on average, teachers in this district tended to reject this view and might have disagreed to an even greater degree if a wider scale had been provided.

Across grade levels, teachers in this school district (n = 105) reflected recent mathematical reform views such as those presented by NCTM (1989, 1991) and discussed earlier. These data are summarized in Table 4. Teachers agreed with the reform-based views expressed in the *invented solution methods are effective* (M = 2.5)

TABLE 4
Teachers' Mean Responses to Various Belief Constructs, and by Grade Level

Construct	*Cronbach's α*	*Elementary*[a]	*Middle School*[b]	*High School*[c]	*All Teachers*[d]
Algebra is best	.76	4.61	4.41	3.98*	4.3
Invented solution methods are effective	.70	2.25	2.43	2.82*	2.5
Symbol-precedence view	.65	3.55	3.42	2.94*	3.3
Teachers should encourage invented solution methods	.84	2.31	2.73*	3.13*	2.7
Product over process	.78	5.38	5.13	4.67*	5.1
Alternative solution methods indicate knowledge gaps	.68	4.95	4.74	4.16*	4.6

Note. Responses given from a scale of 1 (*strongly agree*) to 6 (*strongly disagree*).
[a]$n = 36$. [b]$n = 30$. [c]$n = 39$. [d]$n = 105$.
*$p < .005$.

and *teachers should encourage invented solution methods* constructs ($M = 2.7$). These state that students can develop and apply successful solution methods on their own and that supporting this is a valuable approach for teaching mathematics.

Teachers tended to disagree with those views that challenged reform-based views, such as algebra is best ($M = 4.3$), product over process ($M = 5.1$), and the alternative solution methods indicate knowledge gaps construct ($M = 4.6$). These are constructs that emphasized correct answers over students' reasoning and solution methods and minimized the importance and efficacy of students' invented problem-solving methods. Teachers views on the symbol-precedence approach of problem sequencing fell in the middle of the scale ($M = 3.3$).

An analysis of covariance (ANCOVA) on the mean rating for each of the six constructs was performed using teacher instructional level (elementary, middle, high) as a factor and years of teaching experience as a covariate. From this analysis, all six constructs showed significant differences in the ratings of elementary and high school teachers.

Grade Level Differences

As summarized in Table 4, high school teachers were least likely to agree with reform views expressed in the survey. They were less likely than their colleagues to agree with the view that students can learn effective problem solving on their own (*invented solution methods are effective*), $F(2, 104) = 9$, $MSE = 3.27$, $p < .0002$. Also, although teachers in general strongly disagreed with the product over process view, high school teachers were less likely than elementary school teachers to disagree with the view that the students' answers (i.e., the product) were more important than their problem-solving processes, $F(2, 104) = 9.9$, $MSE = 9.95$, $p < .0001$.

High school teachers also did not give students' invented strategies as much credit as their colleagues in middle and elementary school. Despite the efficacy of students' informal methods, the survey data showed that high school teachers were less optimistic of the successes of students' inventions than were elementary and middle school teachers. Many high school teachers disagreed or strongly disagreed (31%) with statements from the *teachers should encourage invented solution methods* construct that students' invented methods were valid and signaled a conceptual understanding of mathematics. This is compared to 17% of middle school teachers and 2.6% of elementary teachers. This produced a significant difference between high school teachers' responses and those of their colleagues, $F(2, 104) = 14.4$, $MSE = 6.5$, $p < .0001$.

High school teachers were less likely than their elementary and middle school colleagues to disagree with the view that alternative solution methods such as those that are invented or adapted by students are indicators of weak skills or poor conceptual understanding (alternative solution methods indicate knowledge gaps), $F(2, 104) = 21.74$, $MSE = 6.5$, $p < .0001$. Hardly any middle school (3%) and elementary school (0%) teachers agreed with this view held by 15% of the high school teachers in our sample. In a similar manner, high school teachers were significantly less likely than their peers to disagree with the view that algebra is the most effective method for solving problems involving unknown values (algebra is best), $F(2, 104) = 14.3$, $MSE = 3.99$, $p < .0001$. Twenty-three percent of high school teachers agreed with this construct, whereas 17% of middle and 2.6% of elementary school teachers supported this view.

High school mathematics teachers were also more likely than their colleagues to agree with the symbol-precedence view that arithmetic is always easier than algebra, and symbol-manipulation skills were a prerequisite to verbal problem solving, $F(2, 103) = 5.5$, $MSE = 3.9$, $p < .005$ (with one missing value). Many high school teachers (31%) agreed or strongly agreed with the symbol-precedence view, as compared to 10% of middle school and 8% of elementary school teachers.

It is worthwhile to summarize the 30 middle school teachers' responses as well because many early algebra concepts (such as symbolic equations, generalized patterns, slope-intercept form, etc.) are presented at that stage of education. Middle school mathematics teachers generally perceived students' prior mathematical knowledge as potentially very effective. They agreed with the views portrayed in the *invented solution methods are effective* and *teachers should encourage invented solution methods* constructs, and they disagreed with the view expressed by the alternative solution methods indicate knowledge gaps construct. They disagreed that algebra is inherently best as a solution approach (algebra is best) and that answers were more important than the processes that led to them (product over process). Collectively, their perspective on the symbol-precedence view that verbal skills must be based on symbolic ones, and arithmetic reasoning strictly precedes algebraic reasoning was in the middle of the scale. The views of middle school teachers in our sample largely paralleled those expressed by elementary school teachers. The major exception was that middle school teachers were significantly less likely than elementary school teachers to agree with the teachers should encourage invented solution methods view, although they generally agreed with this view.

Teachers' views on the belief instrument largely reflect current reform ideas expressed by the mathematics education community and indicate a basic confidence in many of the reform-based principles for mathematics learning and instruction (NCTM, 1989, 1991, 2000). Elementary school teachers expressed the strongest agreement to principles of student-centered and student-directed learning, whereas high school teachers, although generally in agreement with these views, supported these views least, with middle school teachers falling midway on every construct.

Teachers' Problem Difficulty Ranking

With this foundation of teachers' views established among our sample, along with the variations, we now examine how teachers evaluated the difficulty of certain mathematical tasks for their students. Teachers were also asked to rank order the relative difficulty of problems that implicitly compared arithmetic to algebra problems along one dimension and verbal problems to symbol equations along another dimension. We look at their judgments by comparing arithmetic and algebra problems along one dimension and problem presentation format along a second dimension.

Start-unknown versus result-unknown problems. Table 2 shows the average rank orderings provided by all teachers, and the comparative performance of students on those same problem types. Teachers in all grade levels tended to rank start-unknown (algebra) problems as more difficult than result-unknown (arithmetic) problems. This prediction by teachers is consistent with the student performance data and indicates that teachers are sensitive to the arithmetic–algebra distinction and familiar with its effects on students' problem-solving performance. This finding is also consistent with previous research examining elementary (Peterson, Carpenter, et al., 1989) and high school educators' (Nathan & Koedinger, 2000) beliefs concerning problem difficulty.

Story, word, and equation problems. On average, teachers ranked verbal problems as more difficult than symbolic problems and word-equation problems as harder to solve than story problems (see Table 2). As we will see, the split among teachers along grade-level divisions primarily drives these data.

High school teachers ($n = 39$) tended to rank symbolic equations as easier than verbal problems. As a group, the high school teachers placed arithmetic equations (P6, Table 1) in the easiest rank along with arithmetic word equations (P5). They also ranked algebra equations (P3) at the middle level of difficulty, significantly easier than all of the verbal algebra problems, and even ahead of arithmetic story problems. These judgments are consistent with findings reported elsewhere (Nathan & Koedinger, 2000). However, they are at odds with the student performance data (Table 2, Column 5) that showed that students solved verbal problems far more readily than symbol equations.

The rank orderings produced by elementary teachers generally paralleled the ranking produced by high school teachers, although elementary teachers were more likely to rank problems on the basis of algebraic and arithmetic structure than on presentation format. Arithmetic word equations were considered the easiest, as one elementary teacher mentioned, because "it tells you exactly what to do." They were closely followed by arithmetic symbol equations, which presented the problems in "pure math." Like the high school teachers, the elementary-level teachers regarded algebra equations as easier than verbal algebra problems and ranked algebra word and story problems as most difficult, a judgment that directly contradicted the actual performance of students.

Middle school teachers stood out as a group. They showed their attentiveness to the presentation formats of the problems and were far more likely than their elementary and high school colleagues to place verbal problems in the easiest ranks. As Table 2 shows, arithmetic word equations were considered easiest by middle school teachers, arithmetic and algebra story problems were ranked at midlevel difficulty, and equation solving in both arithmetic and algebra was judged to be the most difficult for students. This rank ordering was consistent with student performance, which placed success with verbal problems above that of symbolic problems, particularly algebra equations.

In fact, middle school teachers gave the closest match to the order of problem difficulty actually attained by students, as measured by the Kendall's rank correlation nonparametric statistic, $\tau(6) = .733$, $p = .034$. The ranking of elementary teachers was marginally predictive of student performances, $\tau(6) = .67$, $p = .06$, whereas the ranking provided by high school teachers was not significantly related to performance difficulty, $\tau(6) = 0$.

From these data, two general conclusions may be drawn. First, regardless of instructional grade level, teachers accurately predict that arithmetic problems will be easier for students to solve than matched algebra problems. Second, teachers generally expect that symbolic problems are easier for students than verbal problems. Third, middle school teachers distinguish themselves as the most accurate in their predictions of student performance, especially in the area of verbal problem solving, whereas high school teachers were the least accurate in judging students' areas of problem-solving difficulty.

Relating Teachers' Beliefs to Difficulty Ranking: A Regression Analysis

Teachers' grade level and response data from the six belief constructs were considered as factors in a regression analysis used to evaluate which factors were reliable predictors of teachers' difficulty ranking. Because teachers were most inaccurate in predicting students' performance on verbal problems relative to symbolic equations, the rankings of these two problem types were of primary concern. Two regressions were performed in this study, one for teachers' ranking of symbolic problems and one for ranking verbal problems, because they actually capture two different dimensions of teachers' responses: teachers' views of symbol-problem difficulty and their views of verbal-problem difficulty. The separate regressions are also needed because selection of a given rank for one problem constrains the possible rankings of the other problems.

The dependent variable for the symbol-problem regression was computed for each teacher from the average ranking of the symbolic arithmetic and algebra problems. The dependent variable for the verbal-problem regression was computed for each teacher from the average ranking of the algebra and arithmetic problems presented in story and word-equation formats. Because the data from the belief constructs were also obtained from multiple measures, composite scores from each of the six belief instrument ratings were also constructed for each teacher and served as predictors. Two additional factors were considered in the regression analysis—years of teaching experience and the grade level taught by the teacher.

The average number of years of teaching experience ranged from 0 to 34 years, with a mean of 14.3 years. However, years of teaching experience proved not to be a reliable factor in any of the analyses, and, therefore, it was removed. Grade level (elementary, middle, and high) proved to be important in predicting teachers' difficulty ranking. Although all of the belief constructs were found to be reliable, only Symbol Precedence emerged as a significant factor for predicting teachers' problem difficulty ranking. Thus, each regression equation used Grade and Symbol Precedence as the two factors to predict teachers' difficulty ranking of either symbolic or verbal problems.

Table 5 shows the relative contribution of each factor in predicting the rank order for verbal and symbol problems. In predicting teachers' rank ordering of symbolic (equation) problems (P3 and P6, Appendix), the combined factors Grade and Symbol Precedence produced a reliable model, $F(2, 96) = 37.7$, $MSE = 45$, $p < .0001$, accounting for 52% of the variance of teachers' difficulty ranking. For predicting the rank ordering of verbal problems, the combined factors Grade and Symbol Precedence again provided a reliable model, $F(2, 96) = 23.3$, $MSE = 10.8$, $p < .0001$, accounting for 39% of the variance.

TABLE 5
Correlations of the Reliable Factors With Teachers' Problem-Solving Difficulty Rank Orderings.

	Factors	
Dependent Variable	*Grade Level* ΔR^2	*Symbol Precedence* ΔR^2
Rank of verbal problems	.26	.13
Rank of symbol problems	.40	.12

Summary

Middle school teachers were most accurate in predicting students' problem-solving performance in contrast to the view held by many high school teachers that symbolically presented problems (i.e., arithmetic and algebraic equations) were easier to solve than verbally presented problems (i.e., story and word equation problems). Teachers' levels of agreement with items consistent with the so-called symbol-precedence view of algebra development proved to be a reliable predictor of teachers' judgments regarding difficulty for both verbal and symbolic problems. This relation, although only correlational, lends support to the hypothesis that the symbol-precedence view mediates teachers' judgments regarding students' mathematical development. In the following discussion we speculate on how teachers' beliefs affect their instructional practices and how these beliefs may shape the learning experiences of children. We will also discuss how the results of this research program can serve the design of a cognitively informed mathematics curriculum for early algebra and the construction of developmentally informed assessment instruments. We conclude with some broad reflections on the study of teaching as a complex cognitive behavior and the role such studies may play in teacher education and fostering change in teachers' beliefs and practices through professional development programs and reform efforts.

DISCUSSION

No pedagogical theory is complete enough to stipulate in advance all of the instructional decisions that teachers face. Systems of beliefs about instruction and student learning are used by teachers to fill in gaps and to organize the complex, dynamic, and uncertain demands of classroom planning and instruction (Eisenhart, Shrum, Harding, & Cuthbert, 1988). Teachers' beliefs about students' abilities play a central role in shaping teachers' judgments and instructional practices (Borko & Putnam, 1996).

Some Factors That Influence Teachers' Judgments

An important finding of this study is that the majority of teachers in our sample generally held reform-based views of mathematical learning and instruction. Teachers generally believed that students can develop and apply effective solution methods and that they should encourage students to adapt and invent methods in service of their problem-solving activities. Teachers also generally believed that there are many ways to solve algebra-level problems, that using alternative methods demonstrates conceptual understanding, and that students' methods, not their final answers, should be principally evaluated.

However, these reform-minded teachers did not seem guided by these particular beliefs when they were judging how students would perform on a set of arithmetic and algebra problems. For instance, almost all of the teachers agreed that student-directed reasoning is an important part of mathematical ability and that students are able to solve algebraic problems using arithmetic and other invented methods. Yet, most teachers did not seem to take those ideas into account when formulating their judgments about the relative difficulty of the given problems for students. Instead, the teachers' rankings exhibited a symbol-precedence view of mathematical development that seems at odds with their view of students' reasoning.

This finding is consistent with research discussed earlier documenting ways that teachers' professed beliefs do not always match their instructional practices (e.g., Borko et al., 1992; Cooney, 1985; Raymond, 1997; Thompson, 1992). For example, Borko et al. (1992) showed that reform-based views of instruction such as the importance of making mathematical procedures relevant and meaningful to students were evident in one teacher's beliefs who strove to provide good explanations and examples that would enable students to "understand the reasoning behind [the procedure] and the logic of it" (p. 204). The failure of the teacher's lesson to impart to the students a conceptual understanding of the invert-and-multiply procedure for division of fractions was attributed to at least two causes. First, the teacher lacked sufficient content knowledge of the conceptual basis for the division rule to impart it to her students ("I don't know why you invert and multiply, I just know that's the rule," p. 207). Second, she exhibited inadequate pedagogical content knowledge in that she did not appreciate the necessity of making this difficult idea meaningful to students because the procedure itself was so simple to apply. The teacher was also unable to construct an adequate way to help confused students represent or visualize the procedure conceptually.

Along with limitations in content and pedagogical content knowledge, our results suggest that an additional factor that may influence teachers' beliefs and their instructional decisions and actions is the way mathematical material is organized in textbooks. This is indicated most strongly by the relation between the survey data and the problem ranking data. Teachers' views of students' learning and performance clustered reliably around all six of the belief constructs in our survey, but only one of these, the symbol-precedence construct, proved to be a reliable predictor of teachers' judgments of problem difficulty. As Nathan and Long (1999) noted in their content analysis of pre-algebra and algebra textbooks, certain untested assumptions about the trajectory of mathematical development appear to be institutionalized in textbooks. Regardless of their validity, these assumptions shape curricula and may be internalized by teachers, ultimately influencing their views of student learning in unproductive ways.

The symbol-precedence view of algebra development presents material in accordance with a task analysis that delineates the component subtasks and learning hierarchies that can inform instructional sequences (e.g., Gagne, 1968). This approach also breaks down the instructional units into manageable sizes for lesson planning. However, task analyses can produce misleading results if studied in isolation from the task context, including the prior knowledge of the student and the larger problem-solving situation itself. The analysis can mischaracterize the task by neglecting the dynamics of the problem-solving process. For example, analyses of students' behaviors have shown that successful informal methods tend to be elicited by verbal problems more often than symbolic problems (Nathan & Koedinger, 2000). Such item effects change the demand characteristics of the task and alter subsequent problem-solving performance. Because of the prevalence of item effects, individual differences, and varying task demands, Glaser (1976) advocated performing detailed analyses of performance as well as of the task itself. Task analyses that do not acknowledge the dynamics of the problem-solving process will look very different than those that do and will lead to very different predictions of future problem-solving behavior.

Textbooks are considered by instructors to be a kind of organizational life raft in a sea of under-specified instructional decisions. However, Borko and Shavelson (1990) cautioned that textbooks and teachers' manuals may actually interfere with teachers' personal decision-making processes and serve as a kind of pedagogical "crutch" for novice teachers. Textbooks may also introduce dissonance when the prescriptions they offer poorly match a teacher's own beliefs about learning and instruction (McCutcheon, 1980). Yet, textbooks and teachers' manuals serve as the major sources of content and instructional activities for many teachers (e.g., Clark, 1978–1979).

It is reasonable to surmise that the use of textbooks in structuring daily classroom lessons, weekly assignments, and year-long curriculum sequencing leads teachers to internalize the images of mathematics they implicitly convey. For example, the beginning high school teacher studied by Cooney (1985) considered problem solving to be at the core of mathematics. To this teacher, the textbook problems were seen as "disguised routinized exercises," and he sought instead to provide "genuine problems that will in some ways excite students and get them involved" (p. 330). Yet, Cooney's detailed interviews and reviews of the teacher's lesson plans revealed that the textbook was the primary influence on the teacher's curriculum as well as on his classroom presentation style.

Grade Level Differences Among Teachers

Although textbooks may be an important factor in shaping teachers' views of mathematics development and instruction, this study also suggests that their influence cannot be considered apart from other factors. The quantitative analyses we present show that inclusion of the symbol-precedence construct accounts for about 12% of the variability in the teacher judgment data, whereas grade-level differences independently account for somewhere between two and three times that.

Why are there pronounced differences between teachers' abilities to predict students' problem-solving performance? Also, why do middle school teachers stand out as particularly accurate in their assessments of early algebra problem difficulty? One reason may be that students at the middle school level have typically not yet received formal algebra training. Middle school students may therefore be more likely than others to use invented methods during classroom assignments to solve start-unknown problems. Middle school mathematics teachers may actually have more opportunities to observe how students make the transition from arithmetic to algebraic reasoning. In contrast, as algebra is typically taught in high school, the focus is often on using formal methods to the exclusion of other methods, invented or otherwise. Therefore, middle school teachers may simply see their students deal with early algebra concepts relatively more often than high school teachers.

Elementary-level mathematics emphasizes the exclusive use of formal representations and solution methods least of all. The elementary school teachers in our study showed the strongest agreement with reform-based views. As with the middle school teachers, it seems likely that they, too, are reasonably familiar with students' use of invented methods to solve problems (e.g., see Carpenter et al., 1988). However, the curriculum does not generally emphasize start-unknown problems. Thus, elementary school teachers are likely to be relatively unfamiliar with students' algebra-level reasoning and may simply expect that by high school these students are engaged in the formal procedures of mathematics the way that it is typically portrayed in textbooks. Because of this, elementary teachers may hold reform-oriented views of learning and teaching and may also operate with a traditional vision of algebra curriculum and instruction.

Data from the belief instruments indicated that high school teachers in our sample—those most centrally charged with algebra instruction—were least aware of the efficacy of students' invented algebra solution strategies. Because high school teachers tend to have greater expertise in their content areas, they are personally more distant from the difficulties of their novice students. This may make high school teachers more susceptible to a kind of "expert blindspot" that prevents them from being made aware of certain aspects of learning such as alternative interpretations of symbolic equations because their pattern-matching skills are so highly tuned (Ericsson & Smith, 1991; Koedinger & Nathan, 1999).

High school teachers were also most likely to agree with statements suggesting that invented methods indicate deficits in students' mathematical knowledge and that algebraic symbol manipulation is the best method for solving problems that deal with unknowns. Teachers who tend to hold formal strategies in such high regard also tend to discount children's mathematical ideas (cf. Carpenter et al., 1989; Fennema et al., 1992). This can alienate children from their own mathematical intuitions. Children in these classes will instead tend to be directed to learn and use abstract, seemingly arbitrary solution methods without bridging them to their own conceptualizations. However, students evidently do not develop their symbol manipulation skills as far as high school teachers hope and believe. Even though there is a great deal of time and attention paid to symbol manipulation, students are still weak in these areas. Additionally, because students are actually less effective at generating correct answers with these formal methods, they will likely experience greater amounts of failure that can lead many students away from mathematics and science (cf. Dweck & Licht, 1980).

Implications for Teacher Decision Making and Instructional Practice

It is important to examine how the symbol-precedence view implicitly advocated by many algebra and pre-algebra textbooks can take root. Consider the case of a student who succeeds in solving a variety of symbol-equation problems at the arithmetic or algebra level. We would expect, in accordance with the student data, that there is a high likelihood for success on comparable story problems. A teacher may interpret this success as support for her view that symbolic-reasoning skills readily transfer to verbal problems. As one teacher in this study put it in her written comments, "Teaching equation solving first provides the student with all the pieces in pure mathematics that he can later apply to word problems." However, our analyses show that this kind of transfer is very unlikely to occur. The better fitting verbal-precedence model of algebra competence (Nathan & Koedinger, 2000) shows that a student who can solve symbolic problems within a level of arithmetic or algebra is further developed mathematically than the one who can only solve story problems. Because symbolic reasoning seems to lag behind verbal reasoning, it is more likely that we are not challenging the student to extend his or her mathematical reasoning when comparable verbal problems are held out as challenge or application problems to a student who routinely solves symbol equations.

If, however, a student fails to solve a symbol-equation problem, a teacher with the symbol-precedence view will likely withhold story problems from the student until the student can demonstrate a certain level of symbolic-skill performance. By making this decision, the teacher may never get to see how the student performs on verbal problems and, thus, never have these assumptions on mathematical development challenged.

Teachers' predictions of students' performances significantly underestimate the difficulty that most students have with symbolic reasoning. In so doing, they appear to pass over some of the conceptual and structural underpinnings of symbolic forms of representation (e.g., Kieran, 1992; Sfard, 1991). Instructional decisions from this perspective may not adequately address gaps in students' understanding of the representational structures and the mathematical procedures that manipulate them.

Perhaps there is, lurking in our data, some support for a holistic view of instruction in which students develop a variety of quantitative reasoning methods, each with their own strengths and demands and use them in a variety of problem-solving settings. At a minimum, these data suggest that mathematics educators need to be more aware of the range, flexibility, and efficacy of students' alternative mathematical problem-solving strategies and the difficulties students have developing their symbolic-reasoning abilities. To support teachers' pedagogical content knowledge in this area, it is natural to recommend that the research community find ways to disseminate its findings on students' reasoning methods. However, a more concerted effort to share findings about the varieties of students' reasoning is not enough. Students' methods may be manifold, and their variability may be extensive. Ultimately, we need to look to teachers to draw out students' alternative-reasoning methods and share them within their professional community.

Implications for Teacher Education

We present evidence that mathematics teachers operate with views of teaching and student reasoning that do not always match student performance. Because these views can have significant impact on teachers' instructional practices and students' learning experiences, it is natural to consider some of the implications of this work for the design of teacher education and professional development programs.

Beliefs and knowledge play a central role in complex cognitive behavior (Newell, 1989; Schoenfeld, 1983). Therefore, the intuition goes, these beliefs must also play a key role in any attempts to change complex behavior. Many approaches for changing deep-seated beliefs about the world recognize the need to explicitly challenge one's initial conceptions. For this reason, belief elicitation is a central part of many programs of conceptual change (e.g., Posner, Strike, Hewson, & Gertzog, 1982) and reform-based professional development (e.g., Fenstermacher, 1994; Nathan & Elliott, 1996; Nathan, Elliott, Knuth, & French, 1997; Richardson, 1994). Once the current beliefs are made overt, a valuable step is to show in some salient way that the current beliefs are inadequate for describing the phenomenon in question. Plans to effect change must also provide a teacher with new beliefs that are accessible, more valid than the old ones, and useful in their teaching.

One way to address the accessibility issue is to identify those aspects of education that teachers already tend to focus on and those they feel most favorable toward. For example, in planning their instruction, teachers focus mainly on subject matter content and instructional activities (Borko & Shavelson, 1990). Student teachers have also reported their most positive feelings toward instructional and in-class activities, especially those that allow them to take responsibility, exercise their own control, and create environments that make student progress salient (Eisenhart et al., 1988). Programs designed to institute teacher change are unlikely to succeed unless they can be made compatible with teachers' existing belief systems (Eisenhart et al., 1988). Consequently, understanding teachers' beliefs about instruction and student learning is essential for instituting changes in teachers' practices, be it for teacher education or for the implementation of reform-based curricula (Fenstermacher, 1979).

For example, in one successful approach (Carpenter et al., 1989), investigators in the Cognitively Guided Instruction program have shown that teachers' instructional practices can be changed by providing them with well-organized information about children's actual thinking and strategy use during simple arithmetic story problem solving. Carpenter et al. chose to focus at the level of students' problem-solving behaviors and performances because this matched teachers' focus on content and instruction most directly.

It is reasonable to employ a similar method when designing a teacher educational program for early algebra instruction that targets inaccuracies in teachers' preconceptions of student learning and development. For example, one may consider changing teachers' beliefs about students' problem-solving performance by inviting teachers to publicly present and defend their rankings for problems like the ones used in this study and then using students' written work and performance data as a way to substantively challenge teachers' intuitions. Although such confrontational approaches may seem appropriate for teacher education, we must acknowledge that the structure provided by beliefs is a major source of any resistance to change. People are generally reluctant to give up their beliefs about important aspects of their lives or their professional practices because of the cognitive disorder that would ensue (Eisenhart et al., 1988).

Establishing the validity of new instructional prescriptions to replace inadequate beliefs poses a significant challenge. Previous efforts at curriculum and instructional reform have fallen short partly because reformers failed to account for the decision-making processes of the teachers implementing the programs (Fennema et al., 1992). Learning theories, no matter how elaborate, are not theories of instruction (Cobb, 1988; Goldman, 1991; M. A. Simon, 1995) and cannot specify all of the aspects of a complex learning setting. Implementation of instructional and curricular goals will nearly always rest on decisions that lie outside of the learning theory, decisions made by curricular designers and instructors based on their own beliefs about student development and instruction

(Clark & Peterson, 1986; Nathan, 1998). As we move closer to a scientific foundation for classroom instruction and teacher education, we must heed these limitations, and acknowledge the significant role of teaching professionals in translating theories of learning into practice and in specifying the myriad details that are necessary to actually teach. The ways in which these details are ultimately addressed is influenced by many factors—the teacher's prior learning experiences, grade level and professional education, the available resources, and beliefs about how students learn. A richer picture of teachers' instructional decision making and practices is sure to emerge as we continue to study the beliefs teachers hold and their influences.

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APPENDIX
The Difficulty Ranking Task Handout Given to Teachers

A SURVEY

Below are 6 problems that are representative of a broader set of problems that are typically given to public school students at the end of an Algebra 1 course—usually 9th grade students. My colleagues and I would like you to help us by answering this brief (5 min) survey. We are happy to share the results we obtain with your class this spring.
What we would like you to do:
Rank these problems starting with the ones you think were easiest for these students to the ones you think were harder. You can have ties if you like. For example, if you think the fourth problem (#4) was the easiest, the 3rd was the most difficult, and the rest were about the same, you would write:

4 (easiest)
2 1 5 6
3 (hardest)

(Feel free to include an explanation of any assumptions you made in the space below.)
Problems:

1. When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour does Ted make?
2. Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?
3. Solve for x: $x \times 6 + 66 = 81.90$
4. When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?
5. Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?
6. Solve for x: $(81.90 - 66) / 6 = x$