REPRESENTATIONAL FLUENCY IN MIDDLE SCHOOL: A CLASSROOM STUDY

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This study investigated seventh- and eighth-grade students' ($N = 90$) abilities to solve problems using tabular, graphical, verbal, and symbolic representations and to translate among these representations before and after instruction. Students experienced more success solving problems using a given representation than translating among representations, with success strongly influenced by the particular representational formats involved. Performance improved with instruction, with the greatest gains occurring with an experimental curriculum (*Bridging Instruction*) that explicitly built on students' invented strategies and representations. Results suggest there are significant gaps between students' abilities to comprehend and to produce representations, that students may attain fluency with instance-based representations (tables and pointwise graphs) before holistic representations (symbolic equations and verbal expressions), and that instruction that explicitly bridges from students' intuitions about quantitative relations can enhance students' abilities to work within and translate among various representations.

One important aspect of mathematical competence is the ability to reason with and among multiple representations. The National Council of Teachers of Mathematics (2000) calls for an increased focus on a variety of mathematical representations, including graphs, tables, symbolic expressions, and verbal expressions, as well as the interconnections among them. These skills, which we call *representational fluency*, are growing in importance as the mathematics education community struggles to reform algebra instruction and curricula.

Despite the documented need for and benefits of representational fluency (e.g., Kaput, 1989), little is known about students' abilities to solve problems presented with different representations or to translate among different representations. This study investigated these issues as well as the impact of a theoretically guided approach for classroom instruction aimed at the development of algebraic reasoning in the middle grades. Specifically, we investigated 4 research questions:

1. How is initial student problem-solving performance influenced by representational format?

2. How is representation use during problem solving influenced by instruction?
3. How well can beginning algebra students initially translate among representations?
4. How is fluency among representations influenced by instruction?

Theoretical Framework

Motivated by the need for sustained, in-school research on students' representation use, this study was carried out in a classroom setting. It compared the influences of a theoretically guided experimental curriculum (described below) and *Connected Mathematics* (CM; Lappan, et al., 1998a, 1998b), a reform-based middle school mathematics curriculum shown to be effective (Hoover, Zawojewski, & Ridgway, 1997). CM emphasizes collaborative, discussion-oriented activities that use data gathering and representation as well as problem solving to make mathematics conceptually meaningful to students. Because it also addresses connections among mathematical representations, and because it was widely used at the school site under investigation, CM served as the control curriculum for this study.

The experimental curriculum implemented in this study, *Bridging Instruction* (BI), grew out of prior research on students' algebraic reasoning that showed the prevalence and power of students' informal and invented problem-solving strategies (Hall, Kibler, Wenger & Truxaw, 1989; Kieran, 1988, 1992; Koedinger & Nathan, 1999; Koedinger, Alibali, & Nathan, 2002; Nathan & Koedinger, 2000a, 2000b; Tabachneck et al., 1994) and their promise for supporting deep, conceptual understanding of more advanced mathematics (e.g., Nathan & Koedinger, 2000c). Like CM, BI takes a collaborative, problem-based approach to mathematics education. However, BI starts by eliciting students' invented strategies and representations and explicitly *bridges* from these to more formal and more efficient solution methods during classroom discussion. Thus, students' intuitive notions of how to organize data, how to depict them pictorially, and how to describe both linear and nonlinear relationships in words serve as precursors to their use of tables, graphs, and equations, respectively.

Method and Data Sources

Ninety students in four combined 7th/8th grade mathematics classrooms in a middle/upper-middle class school district in the Midwestern U.S. participated in this study for 9 weeks. Two of the classrooms were designated control classrooms and implemented CM, the school's standard curriculum, while the other two classrooms implemented the experimental curriculum, BI. The same, regular classroom teacher taught all four classes.

The CM curriculum is well described (Lappan, Fey, Fitzgerald, Friel & Phillips, 1998a, b) and commercially available. Students in the CM (control) condition worked through the seventh-grade units *Variables and Patterns* and *Moving Straight Ahead* during the time of this study.

The BI curriculum approached bridging in several ways. First, students' invented solution strategies and representations served as conceptual bridges to more formal procedures and representations. In prior studies (e.g., Nathan & Koedinger, 2000c) middle school students were observed using various invented solution methods to solve story problems and systems of equations. Guess-and-Test is one such method that students intuitively apply. It serves as a powerful inroad into formal algorithms because it highlights the structural aspects of the algebraic relations (Kieran, 1992) and procedurally grounds the concept of variable. Throughout the 9-week intervention, BI students were called upon to first decide for themselves, individually or in small groups, how they would represent data or solve for unknown values.

For example, in a variant of the "bridges and pennies" task (week 1; adapted from CM 8th grade series), students were asked to make a table to record the thickness of a paper bridge and the number of pennies that just breaks that bridge. They were then given wide latitude to graph the relationship between bridge thickness and number of pennies. Many insights emerged about students' understandings of graphical representations from the public displays of their graphs. Students did have a general understanding of the need for axes, labels, and plotted points, but did not all agree on how to label axes. Discussions of scale and interval size, and of the nature of dependent and independent variables naturally emerged. Students also did not all treat the independent variable (bridge thickness) as continuous. Many groups only included the values for which there were data collected, treating bridge thickness initially as nominal. This led to important discussions of the similarity of an axis to a number line. Finally, when students were asked to interpret their graphs, many read off relations like one would read values from a table. This led to the idea that one could use a line of best fit instead of the individual points to find an overall relationship between bridge thickness and bridge strength (measured in pennies).

In a second aspect of bridging instruction, concrete analogs served to ground more abstract representations. For example, in the "building squares" activity (week 3), students investigated the relationship between the side length and area of a square by building squares out of 1-inch square tiles. To see how area grows as a function of side-length, students used a method described by Kalchman (1998). Students decomposed the squares, stacked the tiles along grids of 1-inch graph paper, and marked off the values (heights) that showed the number of square-inches covered by each square. This provided a concrete way for students to both record the area of each square and to see how that area could be represented as the height along an axis of a graph.

In a third aspect of BI, linear and nonlinear relations served as contrasting cases for one another in order to make properties of each more salient to learners. In "building squares," students were explicitly asked to observe how both area and total tile perimeter (the sum of the side-lengths of all the tiles) grew as a function of the side-length of the squares, and to describe verbally and visually how these two growth patterns were similar and different. This idea was expanded further in "the cube problem" (weeks 8 and 9) in which the growth patterns for corner pieces (constant function), edge pieces (linear), face pieces (quadratic), and hidden pieces (cubic) were described and compared visually, verbally, and symbolically as functions of the side-length of the entire cube. One reason to use contrasting cases during instruction is to help learners form differentiated knowledge structures (e.g., D. Schwartz & Bransford, 1998). Linear functions are often taken as prototypical exemplars of functions. This can have detrimental effects on students' learning because linear functions have some unique attributes that do not generalize to all mathematical functions, such as (a) linear functions form a straight line when graphed against linear axes; (b) only two points are needed to determine a linear function; (c) the rate of change is constant; (d) missing values can be determined through linear interpolation and extrapolation (B. Schwartz & Hershkowitz, 1999). Most students, in fact, over-generalize the properties of linear functions and make erroneous assumptions on the basis of these over-generalizations, such as believing that only one function (a line) can pass between two given points. BI students directly confronted some of these misconceptions by simultaneously considering linear and non-linear functions throughout their algebra unit.

Classroom instruction and student interactions will be described in detail in future reports. Here we report on students' abilities to work within a given representation and to translate among representations, as assessed before and after the two different 9-week instructional interventions. The assessment instrument used a factorial design to allow systematic examination of the effects of problem linearity (linear or exponential), slope-sign (increasing or decreasing function), input representation (graph, symbolic, or word expression), and input-to-output translation (a graph, symbolic, or word expression input paired with a graph, table, symbolic, or word expression output) on student performance.

All problems given in the assessments were first introduced in words. Problems then presented the "input" representation (i.e., a graph, symbolic, or word expression) and asked students to respond to two problem-solving items and one translation item. The problem-solving items required students to work within the given representation to find a specific value of the dependent variable given a specific value of the independent variable or vice versa. The translation items required students to represent the presented functional relationship using a different output representation. Appendix 1 illustrates the multiple forms a linear problem could take given the variety of input and output representations involved.

Results and Discussion

Students' Initial Problem-Solving Performance

Pretest results indicate that linearity and mathematical representations do indeed influence students' problem-solving performance (see Figure 1). Problem-solving performance using graphical representations greatly exceeded that of all other representations. This graphical advantage held regardless of slope-sign (i.e., increasing or decreasing) for both linear (M = 86.0%, SD = 12.7%) and nonlinear (M = 85.3%, SD = 12.2%) functions.

In addition, students performed better on linear (M = 58.3%, SD = 32.1%) than nonlinear (M = 39.7%, SD = 36.0%) problems, $F(1, 68) = 47.50$, $p < 0.0001$. The interaction of linearity and representation was also significant, $F(2, 28) = 20.76$, $p < 0.0001$. Students experienced more success on linear problems when provided a verbal representation than when provided a symbolic one (linear word: M = 65.1%, SD = 22.3%; linear symbolic: M = 24.0%, SD = 21.0%), whereas symbolic representations led to greater success than verbal representations on nonlinear problems (nonlinear symbolic: M = 20.8%, SD = 19.8%; nonlinear word: M = 13.0%, SD = 12.3%; see Figure 1). This replicates the complexity-representation interaction reported by Koedinger et al. (2002) showing that verbal representations are most effective when solving lower complexity problems, whereas symbolic representations are more effective for higher complexity problems.

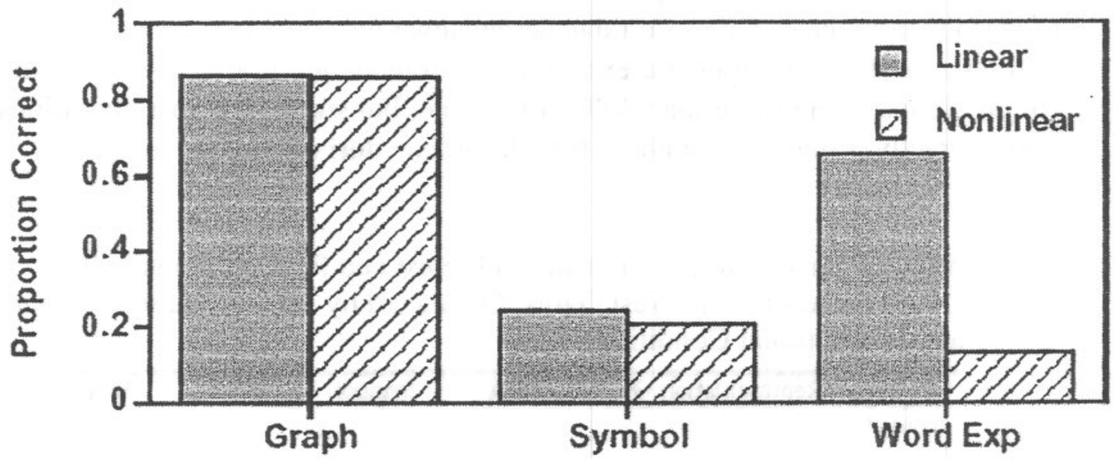

Figure 1. Proportiona of problem-solving pretest items solved correctly by problem representation and linearity.

Influences of Instruction on Problem Solving

BI students made greater gains in problem-solving performance from pretest to posttest than CM students (see Table 1), yielding a significant date by condition interaction, $F(1, 68) = 4.01$, $p = 0.05$, with CM students' improvements limited to linear functions presented symbolically. Negative CM gains were not statistically different from zero. In contrast, BI gains were distributed across all representational formats and all levels of linearity.

Table 1. Proportion of Problem-Solving Pretest and Posttest Items Solved Correctly Along With Test Gains, Organized by Representational Input, Linearity, and Instructional Condition

Input Representation →		Graph		Symbol		Word	
Instruction ↓		Linear	Nonlinear	Linear	Nonlinear	Linear	Nonlinear
Control (CM)	Pretest	0.85	0.87	0.29	0.24	0.72	0.17
	Posttest	0.77	0.77	0.47	0.28	0.61	0.15
	Gain	-0.08	-0.10	0.18	0.04	-0.11	-0.02
Experimental (BI)	Pretest	0.87	0.84	0.19	0.18	0.58	0.09
	Posttest	0.88	0.90	0.39	0.25	0.67	0.32
	Gain	0.01	0.06	0.20	0.07	0.09	0.23

Students' Initial Translation Performance

Translation among representations appears to be a very advanced skill. At pretest, translation performance was essentially at zero for all but a few types of problems (see Table 2). Students could use pre-constructed input representations to solve problems (Table 1) far better than they could generate new representations as a part of a translation task. In the most dramatic example, pre-constructed graphs were correctly used for problem solving more than 80% of the time, but students across conditions could only correctly produce them about 6% of the time during translation tasks.

Table 2. Proportion of Translation Pretest and Posttest Items Solved Correctly Along With Test Gains, Organized by Representational Input and Instructional Condition

Output Representation →		Graph	Symbol	Table	Word
Instruction ↓					
Control (CM)	Pretest	0.11	0.19	0.11	0.02
	Posttest	0.10	0.10	0.34	0.05
	Gain	-0.01	-0.09	0.23	0.03
Experimental (BI)	Pretest	0.00	0.12	0.12	0.06
	Posttest	0.30	0.13	0.38	0.21
	Gain	0.30	0.01	0.26	0.15

Influences of Instruction on Representational Fluency

Overall, students were more successful on the translation items at posttest than at pretest, $F(1, 68) = 42.86$, $p < 0.0001$, though post-instruction performance was still low in both conditions (Table 2). BI students made greater translation gains than

CM students, $F(1, 68) = 5.94$, $p = 0.02$. The CM group experienced a 5.6% increase in correct responses, while the BI group experienced a 17.6% increase. Negative CM gains were not statistically different from zero. By posttest, CM students could translate from any input representation to a table of values at a rate significantly greater than zero while BI students could translate from any of the input representations to a table of values and to a graph, and could also translate between word expressions and symbolic equations.

Students understood and produced instance-based representations (i.e., tables and point-wise graphs) far better than more holistic representations (symbolic and verbal expressions, and line graphs). Graphs present an interesting case within the instance-based/holistic dimension, because they can be either instance-based (as with scatter plots and bar graphs) or holistic (as with line graphs). This duality does not apply to symbolic expressions, word expressions, or tables of values. To further understand students' performance with representations along this dimension, we compared experimental students' post-intervention abilities to produce accurate graphs when they were judged with instance-based versus holistic scoring criteria. An instance-based graph was defined as one including at least three correct data points. It was not necessary for the axes to be labeled with words or for the points to be connected with a line or exponential curve. Holistic graphs were ones in which the function (line or exponential curve) was drawn and the correct y-intercept was included. When evaluated from the point-wise perspective, students exhibited relatively high levels of performance (M = 29.5%). When they were evaluated using holistic criteria, performance was much lower (M = 4.3%). However, even when students constructed instance-based graphs, they often made other types of errors (see Figure 2).

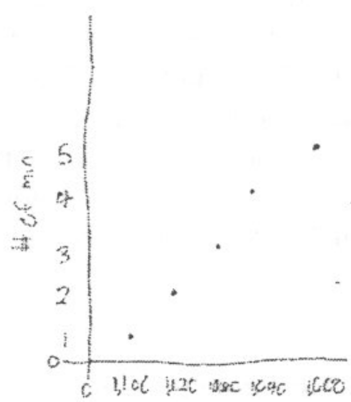

Figure 2. Point-wise graph of function d = 1200 - 40n

Conclusions

This study documents students' pre-instructional knowledge of mathematical representations and the effects of instruction on representational fluency. Students' pre- and post-instructional problem solving was heavily influenced by representational format, with students succeeding much more often with graphical representations than with symbolic or verbal representations, for both linear and nonlinear problems. On linear problems, students succeeded more often with verbal representations than with symbolic representations, and on nonlinear patterns, this pattern was reversed. This finding replicates earlier research showing a trade-off among representations, such that verbal representations are more effective for simpler problems, whereas symbolic representations are more effective for more complex problems (Koedinger, et al., 2002).

Our findings are consistent with Kalchman, Moss, and Case's (2001; Kalchman, 1998) psychological theory of the development of children's understanding of mathematical functions, which holds that procedurally based (i.e., computational) representations (e.g., tables of instances) and analogical representations (e.g., bar graphs) developmentally precede and form the basis for the more integrative, holistic representations (e.g., line graphs). In the translation tasks used in this study, students were most successful at generating tables of values. They were also more successful at generating instance-based graphs (e.g., scatterplots) than at generating holistic graphs (e.g., line graphs).

Pedagogically, one plausible hypothesis is that tables and instance-based graphs are a natural entry point into the mathematics of covariation, which serves as a central idea for the mathematics of functions. Because of their dual nature, graphs may be particularly effective in helping students to bridge from instance-based to more holistic representations. Ultimately, they may help students to learn the more holistic formalisms and to reap the rewards of such formalisms in the face of increasingly complex (e.g., nonlinear) relationships.

This study also demonstrated that *Bridging Instruction* was effective at facilitating both problem solving and translation. Why was *Bridging Instruction* so successful? One possibility is that the approach provides conceptual grounding for the meanings of various algebraic representations, by explicitly connecting them to students' informal reasoning and their intuitions about data organization and quantitative relations. This conceptual grounding may serve to support representational fluency. Another possibility is that it is pedagogically valuable to address linear and nonlinear functions within the same instructional unit. Each type of function can serve as a contrasting case for the other, reinforcing important common concepts while highlighting important distinctions. The present study does not allow for definitive conclusions about the individual or combined effects of bridging from students' informal reasoning or of teaching linear and non-linear functions in tandem. Future studies of each of these factors will be needed to tease out the basis for the effects we have observed. However, at a minimum, the present work documents that an instructional approach grounded in students' informal reasoning, and focused on both linear and nonlinear functions, can be highly effective and suggests that such instruction deserves greater consideration in the classroom.

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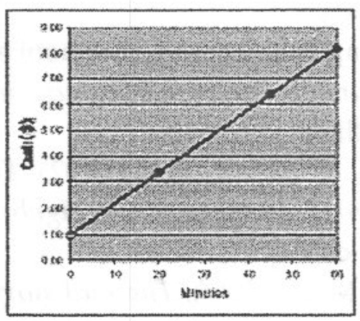
Appendix

Examples of Multiple Forms of a Linear Problem

Problem Section	Information Presented
Situation introduced	Cassandra sells phone cards to college students so they can make long distance calls for a good price. Each card has a base charge and a per-minute rate.
Input presented (For a given problem students get only one of these three input representations)	*Graph Input*: Below is a graph you could use to find the price of the card if you know the number of minutes on it.  *Symbol Input*: The expression below shows how to find the price of the card, p, of you know the number of minutes on it, n. $p = 0.99 + 0.12n$ *Word Expression Input*: The description below tells you how to find the price of the card if you know the number of minutes on it. To find the price of the card, you multiply the number of minutes by the per-minute rate of $0.12, and then add the base charge of $0.99.
Part a (problem solving)	What would be the price of a card with 30 minutes?
Part b (problem solving)	How many minutes would be on a card that cost $6.99?
Part c (translation) (For a given problem students are asked to translate to only one of these three output representations)	*Graph Output*: Make a graph that you could use to find the price of the card if you know the number of minutes. (*Examination packets had graph paper included.*) *Symbol Output*: Write a mathematical expression that tells how to find the price of the card if you know the number of minutes. *Table Output*: Make a table of values that you could use to find the price of the card if you know the number of minutes. *Word Expression Output*: Describe in words how to find the price of the card if you know the number of minutes.