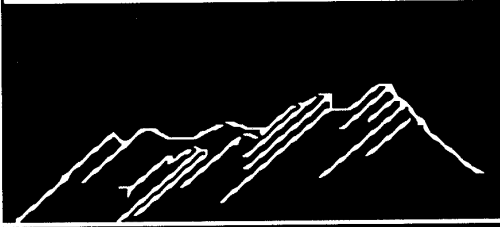


Institute of Cognitive Science



# Technical Report

University of Colorado, Boulder

## Enhancing Middle School Students' Representational Fluency: A Classroom-Based Study

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RUNNING HEAD: Representational Fluency

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Enhancing Middle School Students' Representational Fluency:

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RUNNING HEAD: Representational Fluency

## Abstract

The present study investigated middle school students' abilities to work within and translate among tabular, graphical, verbal, and symbolic representational formats. Seventh and eight grade algebra students were more successful at solving problems using a representation than at translating among representations. However, performance was strongly dependent on the particular representational formats used. Student performance improved with instruction, and the greatest gains were found for an experimental curriculum (*Bridging Instruction*) that focused on both linear and nonlinear relations, and made explicit links to students' invented strategies and representations. Students' performance suggests that there are often significant gaps between their abilities to comprehend and to produce representations. Finally, students appear to attain fluency with instance-based representations (such as tables and point-wise graphs) before holistic representations (such as symbolic equations and verbal expressions).

One important aspect of mathematical competence is the ability to reason with and among multiple representations. The importance of this skill, which we call *representational fluency*, is becoming increasingly recognized as the mathematics education community struggles to reform algebra instruction and curricula. In the *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (2000) acknowledges that there has historically been a “preoccupation with number” (p. 211) and calls for an increased focus on a variety of representations, including graphs, tables, symbolic expressions and verbal expressions, as well as the interconnections among them.

Members of the research community have also trumpeted both the need for and the benefits of making these connections (e.g., Ainsworth, 1999; Brenner, et al., 1997; Confrey & Maloney, 1996; Dossey, 1997; Knuth, 2000). For example, Knuth (2000) showed that advanced high school algebra students who are familiar with equations and graphs do not readily connect graphical representations such as the Cartesian coordinate system to their knowledge of equations, and fail to use graphs even when graphical solutions are easier and more efficient. He suggests this may be due to an almost exclusive curricular and instructional focus on symbolic representations and manipulations. Ainsworth (1999), in his analysis of types and trade-offs of multiple representations, argues that translation across representations should be supported because it can maximize learning outcomes. Kaput (1989) suggests that mathematical meaning making is actually built upon the ability to translate within and among various representations, and that fundamentally, meaning is based on a “relational semantics” between “linking representations” including internal mental representations and physical systems as well as tables, symbols, and graphs (p. 168). Moschkovich, Schoenfeld, and Arcavi (1993) argue that competence in the mathematics of functions depends upon moving flexibly among representations. They go so far as to use the criterion of connections among representations to evaluate curricula and student assessments. At the same time, they document just how

rare and difficult this level of competence is among students.

Research from a psychological perspective has also shown important benefits of multiple representations for learning and performance. For example, Tabachneck and colleagues (1992; Tabachneck, Leonardo, & Simon, 1994, 1995) showed how an expert in economics achieved an understanding of an economic situation that was thought to be out of the reach of novices by combining graphical and verbal representations. Schwartz (1995) found that the availability of multiple representations played a key role in students' generation of abstract representations. Dyads of students working together were more likely to generate abstract representations than were matched individuals working alone, and Schwartz attributed this difference to their need to communicate across the multiple representations generated by the dyad members. Evidence from both behavioral research (e.g., Griffin, Case, & Siegler, 1994; Stenning & Oberlander, 1995) and neuroscience (e.g., Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999) point to a dual system of linguistic and spatial representations that supports mathematical reasoning. Still, psychological models of how people integrate multiple representations in support of problem solving reveal that this process can be very cognitively demanding (e.g., Stenning & Oberlander, 1995; Tabachneck, et al., 1994, 1995).

### *Focus of Research*

Despite the perceived and documented benefits of representational fluency, little is known about the various performance tradeoffs for different representations used by mathematics students, or about how well students move among various representational formats during problem solving. Prior studies of these questions have had some key limitations, emphasizing study of mature students and adults, typically in laboratory and "pull-out" settings (see Grouws, 1992, for review). One important exception is an in-class, one-month intervention conducted by Brenner and her colleagues (1997) with English and Spanish-speaking students. The intervention emphasized translation among different

representations (tables, graphs, words, equations) as students discussed, represented, and solved problems with meaningful contexts in a collaborative, guided discovery setting. Students in the treatment condition improved more than comparison students on a variety of word problem solving and representational use measures, regardless of ESL status.

Motivated by the promise of studies such as Brenner and colleagues (1997) and the need for more sustained, in-school research on students' representational use, the current study was carried out in a classroom setting. It addresses two sets of questions, described below, about the performance of middle school students in mixed 7<sup>th</sup>/8<sup>th</sup> grade classes using and translating among various representations (words, tables, graphs, and algebraic symbols) as they reason about quantitative problems. For each set of questions, we investigated the impact of a theoretically-guided approach for classroom instruction lasting 9 weeks and taught by the regular classroom teacher that is aimed at the development of algebraic reasoning prior to formal, pre-algebra instruction.

*Working with representations in support of problem solving*

- How is student problem-solving performance influenced by representational formats?
- How is representation use enhanced by instruction?

*Translating among representations*

- How well can early algebra students translate among representations?
- How is fluency among representations enhanced by instruction?

These four questions were studied with respect to two curricula. *Connected Mathematics — Seventh Grade Series* (Lappan et al., 1998a, 1998b) was the standard curriculum for the schools, and so served as the control condition. This reform-based curriculum takes a conceptual approach and emphasizes collaborative, discussion-oriented

activities that use data gathering and representation as well as problem solving to make mathematical representations and procedures meaningful to students. Connected Mathematics (CM) is widely used in the U.S. and has been shown to be an effective curriculum for teaching middle school mathematics (Hoover, Zawojewski, & Ridgway, 1997).

The alternative curriculum, *Bridging Instruction* (Koedinger & Nathan, 1999; Nathan & Koedinger, 2000c), also takes a collaborative, problem-solving based approach to develop a conceptual understanding of mathematical representations and procedures. However, Bridging Instruction (BI) starts with students' informal approaches to problem solving and translation tasks. These intuitive notions serve as the foundation for classroom teaching and discussion that help to conceptually bridge between students' informal mathematical ideas and the target, formal representations and procedures. For example, the iterative nature of the guess-and-test approach that students commonly invent to solve algebra problems before they have had algebra instruction serves as a starting point to introduce the notion of *variable*.

The BI method takes an approach that is highly inductive and reflective of students' ideas and problem-solving processes. BI grew out of research on students' pre- and post-instructional algebraic reasoning that showed the prevalence and power of students' invented solution strategies and representations that came from outside of their formal instruction (Hall et al., 1989; Kieran, 1988, 1992; Koedinger & Nathan, 1999; Koedinger, Alibali & Nathan, 2000; Nathan & Koedinger, 2000a, 2000b; Tabachneck et al., 1994). Results from some of these earlier studies challenged deep-seated views of instruction and curricular sequencing, such as the idea that instruction in symbolic problem solving should precede word problem solving (e.g., Nathan & Koedinger, 2000b; Nathan, Long, & Alibali, in press). To date, BI has been used with a small number of 6<sup>th</sup>, 7<sup>th</sup>, and 8<sup>th</sup> grade students in previous teaching experiments to study its impact on student performance and on teachers' understandings of student reasoning and development (Masarik & Nathan,



2000; Nathan, 1999). Generally, bridging from students' intuitive conceptions of quantitative relations to more formal representations proves to be a powerful and practical way to make symbolic representations and procedures meaningful to students (Nathan & Koedinger, 2000c).

## Method

### *Participants*

Ninety 7<sup>th</sup> and 8<sup>th</sup> grade students in four mathematics classrooms in a middle/upper-middle class school district in the Midwestern U.S. participated in this study. Eight students were excluded from the final analyses because they switched classrooms in a way that altered their condition assignment during the course of the study. Of the remaining 82 students, 70 took both the pretest and posttest assessments and therefore are included in all analyses that involve participants as the unit of analysis. An additional 12 students took only one of the two assessments and were absent for the other. Data from these 12 students is included in analyses that involve items as the unit of analysis. Thus, the item analyses are based on the performance of 82 students.

Two of the classrooms were designated control classes and implemented the *Connected Mathematics* seventh grade curriculum (Lappan et al., 1998a, 1998b), while the other two classrooms were designated experimental classes and implemented the *Bridging Instruction* approach described in the introduction. The same teacher taught all four classes. All were mixed 7<sup>th</sup>/8<sup>th</sup> grade classes. The proportion of 7<sup>th</sup> graders was not identical across conditions (CM: 70.3% 7th graders; BI: 60.6% 7th graders); however there was no significant difference in performance between 7<sup>th</sup> and 8<sup>th</sup> graders either at pretest or posttest, and thus grade level was not considered as a factor in the analyses.

*Materials*

The materials used in this study consisted of an assessment instrument that was given twice to students, before and after intervention, and the curriculum, which varied for the Connected Mathematics (CM) and Bridging Instruction (BI) conditions.

*Assessment design.* The assessment instrument used a factorial design to allow systematic examination of the effects of problem linearity (linear or nonlinear), slope-sign (increasing or decreasing function), input representation (graph, symbol, or word expression), and (for translation items) input-output pair (a graph, symbol, or word expression input paired with a graph, symbol, table, or word expression output) on student performance. For each linearity-slope-sign pair (linear increasing, linear decreasing, nonlinear increasing, and nonlinear decreasing), two “cover stories” were developed. For example, all linear increasing problems were presented either in the context of Luke riding a scooter and keeping track of the distance he traveled, or Cassandra selling phone cards and keeping track of the relationship between the cost of the card and the number of minutes available on the card. Nine problems were constructed for each of the eight cover stories resulting in a total of 72 different problems.

These 72 problems were distributed among twelve different forms with three linear and three nonlinear and three increasing and three decreasing problems per form. These twelve forms were broken into six pairs, with forms in each pair having identical mathematical structure but different cover stories. For example, one form presented a linear, increasing, graph input-table output problem using the “scooter” cover story while its pair presented a linear, increasing, graph input-table output problem using the “phone card” cover story.

All problem situations were first introduced in words. Problems then presented the “input” representation (i.e., a graph, symbolic equation, or word expression) and asked students to respond to three items. On linear problems, the first item asked students to use the input representation to find a specific value of the dependent variable given a specific

value of the independent variable, while the second item asked students to find a specific value of the independent variable given a specific value of the dependent variable. On nonlinear problems, the first item asked students to use the input representation to find a specific value of the dependent variable given a specific non-zero value of the independent variable, while the second item asked students to find the value of the dependent variable when the value of the independent variable was zero. In both cases these first two items are considered *problem-solving* items. Regardless of problem linearity, the third item asked students to represent the underlying functional relationship expressed in the problem using a representation different from the one initially presented. This was considered a *translation* item. Appendix A provides a detailed look at the linear and nonlinear problems presented to students and the many forms that they took when different input-output pairs were considered.

*Curriculum differences.* The dimensions used to compare the curricula are shown in Table 1. The *Connected Mathematics* curriculum was used regularly by the mathematics teacher, and so served as the control curriculum. CM emphasizes how words, tables, graphs and algebraic symbols can depict data for linear relations, and how these representations are interrelated. Across all of its strands, the CM curriculum emphasizes “moving flexibly between graphic, numeric, symbolic, and verbal representations and recognizing the importance of having various representations of information in a situation” (<http://www.math.msu.edu/cmp/Process.html>). The Algebra strand, one of the four that specify the curriculum content across the grade levels, targets, among others, the following four competencies (<http://www.math.msu.edu/cmp/Strand.html>):

- Using tables, graphs, symbolic expressions, and verbal descriptions to describe and predict patterns of change in variables.
- Recognizing, in various representational forms, patterns of change associated with linear, [and by 8<sup>th</sup> grade] exponential, and quadratic functions.

- Using numeric, graphic, and symbolic strategies to solve problems involving linear [for 7<sup>th</sup> grade], exponential, and quadratic functions [by 8<sup>th</sup> grade].
- Solving linear equations and [by 8<sup>th</sup> grade] simple quadratic equations by manipulating symbols and by using tables and graphs.

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Place Table 1 about here

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Participants in the experimental condition used the *Bridging Instruction* (BI) approach (Nathan & Koedinger, 2000c). Like CM, BI emphasizes representational fluency. However, as described in the Introduction, BI distinguishes itself from CM in that the teacher explicitly draws on students' intuitive mathematical notions and invented solution strategies and representations as starting points for instruction. Thus, students' intuitive notions of how to organize data, how to depict it pictorially, and how to describe both linear and nonlinear relationships among variables served as the precursor to the use of tables, graphs, and equations, respectively. Nonlinear relations addressed both quadratic and exponential functions. Quadratic functions were taught using Kalchman's method of building graphs by recomposing manipulatives (1998; Kalchman, Moss & Case, 2001) and was based on students' grounded notions of area. Exponential relations were taught in the context of splitting and doubling, based on work by Confrey and Smith (1995).

#### *Procedure*

Students in all four classes were given the written assessments at pretest and, approximately 10 weeks later, at posttest. Each assessment form consisted of six problems that tested students' problem-solving abilities (solving for an unknown value) and their abilities to translate from one mathematical representation to another. The regular classroom teacher administered the assessment instrument during the normal time scheduled for

students' mathematics class. Students were given 35 minutes to complete the examinations and were allowed to use calculators.

### *Scoring and Coding*

*Problem Solving.* As noted, the problem-solving portion of the assessment presented students with a representational input (a graph, symbolic equation, or word expression) and asked them to use this representation to find a specific value of the dependent variable given a specific value of the independent variable and vice versa. Given a symbolic equation or word expression as input, answers were coded as correct only if they were exact, with the exception of one problem for which the story context permitted rounding. Given a graph as input, answers were coded as correct if they fell within  $1/8$  inch of the algebraically correct value on the coordinate system. Such error was considered acceptable because students were not given corresponding algebraic equations along with the graphs and thus had to rely solely on the graphs themselves to find solutions.

*Translation.* As noted, translation problems asked students to create a representational "output" (a graph, symbolic equation, table, or word expression) containing information mathematically equivalent to that given in the "input" representation presented in the problem-solving portion of each problem. Given the diverse nature of the outputs, separate criteria for correctness were established for each.

*Graphs.* To qualify as correct, graphs needed to have at least three correct data points and no incorrect data points. It was not necessary for the axes to be labeled with words or for the points to be connected with a line or curve.

*Symbols.* To qualify as correct, symbolic equations needed to be completely accurate. It was acceptable for words to be used in place of variables (e.g., distance = 8 x min).

*Tables.* To qualify as correct, table outputs needed to have at least three accurate entries (ordered pairs) and no inaccurate entries. These entries needed to be exact with the

exception of those generated from graphs, for which the previously described 1/8-inch rule was employed. It was not necessary for table columns to be labeled.

*Word Expressions.* To qualify as correct, word expressions needed to accurately describe the computational strategy necessary to solve the given problem (either entirely in words or in a word-symbol combination).

## Results

The results are divided into two main sections. The first section presents pretest and posttest results for the problem-solving portion of the study, and the second section does the same for the translation portion of the study.

The data were analyzed in two ways: by participant and by test item. Item analyses considered the following main effects and their interactions: condition, date (i.e., pretest versus posttest), linearity, slope-sign, and representation. Because representation and slope-sign were not balanced across participants, the participant analyses considered only the main effects of condition, date, linearity, and their interactions.

### *Problem Solving*

*Pretest performance.* Students in the control group performed slightly better on problem-solving items at pretest than students in the experimental group: (Control  $M = 52.2\%$ ,  $SD = 34.3\%$  vs. Experimental  $M = 45.9\%$ ,  $SD = 36.1\%$ ). This difference in performance was not significant in the participant analysis,  $F(1, 68) = 2.63$ ,  $p = 0.11$ , though it was significant in the item analysis,  $F(1, 12) = 5.83$ ,  $p = 0.03$ .

Both linearity and representation influenced students' success on the problem solving tasks. Students performed better on linear problems than nonlinear problems (linear  $M = 58.3\%$ ,  $SD = 32.1\%$  vs. nonlinear  $M = 39.7\%$ ,  $SD = 36.0\%$ ),  $F(1, 68) = 47.50$ ,  $p < 0.0001$  across participants, and  $F(1, 12) = 10.07$ ,  $p = 0.01$  across items. Students succeeded more often when the given representation was a graph than when it

was a symbolic equation or word expression, regardless of linearity,  $F(2, 12) = 95.43$ ,  $p < 0.0001$ . Slope-sign (increasing versus decreasing) did not significantly influence students' problem-solving performance.

In addition to the main effects of linearity and representation, the interaction of these factors was also found to be significant,  $F(2, 12) = 18.62$ ,  $p < 0.0001$ . As shown in Figure 1, linearity did not influence student performance when problems were presented in graphs or symbols ( $M = 86.0\%$ ,  $SD = 12.7\%$  on linear graph problems and  $M = 85.3\%$ ,  $SD = 12.2\%$  on nonlinear graph problems;  $M = 24.0\%$ ,  $SD = 21.0\%$  on linear symbol problems and  $M = 20.8\%$ ,  $SD = 19.8\%$  on nonlinear symbol problems). However, for problems that were presented in words, students succeeded more often on linear problems ( $M = 65.1\%$ ,  $SD = 22.3\%$ ) than non-linear problems ( $M = 13.0\%$ ,  $SD = 12.3\%$ ). Examining this interaction from a different perspective, we found that students succeeded more often on linear problems that were presented in words than on those that were presented in symbols, whereas on nonlinear problems the reverse was found to be true. These results lend support to previous findings that document a verbal advantage for problems of relatively low complexity, but a symbolic advantage for problems of relatively high complexity (Koedinger, Alibali, & Nathan, 2001). This study showed that, given a fairly simple linear problem situation, students succeed more often when the problems are presented verbally (e.g., as word expressions or story problems) than when they are presented in symbols. However, for more complex problems, students perform better when problems are presented symbolically (e.g., as algebraic equations). The present results replicate the earlier findings. They also extend the results to include graphical representations, suggesting that, like tables, graphs are highly reliable for low-complexity tasks, and, like equations, graphs scale nicely as complexity increases. This dual nature of graphs is elaborated upon in the Discussion section.

*Problem solving gains.* Considering the pretest and posttest data together, we again found significant effects of linearity, representation, and the linearity by representation

interaction. Because these factors were significant at pretest as well, we focus in this section on students' problem-solving gains from pretest to posttest, considering this main effect of date and its interactions with condition, problem linearity, and problem representation.

As expected, students performed better at posttest than at pretest,  $F(1, 68) = 11.62$ ,  $p = 0.001$  across participants and  $F(1, 12) = 4.99$ ,  $p = 0.05$  across items (Pretest  $M = 9.0\%$ ,  $SD = 35.2\%$ ; Posttest  $M = 53.7\%$ ,  $SD = 29.9\%$ ). However, this main effect of date was qualified by significant interactions with both problem representation and condition. Students made greater pre-to-post gains on symbolic problems than on graph and word expression problems, as revealed in the problem representation by date interaction,  $F(2, 12) = 6.374$ ,  $p = 0.01$ . It should be noted that there was little room for improvement on graph problems, as pretest scores were already quite high ( $M = 85.7\%$ ,  $SD = 12.3\%$ ; see figure 1).

Pretest to posttest gains also varied by condition. Students in the BI group made greater gains from pretest to posttest than did students in the CM group (see figure 2),  $F(1, 68) = 4.01$ ,  $p = 0.05$  across participants, and  $F(1, 12) = 9.478$ ,  $p = 0.01$  across items. In fact, performance in the control group actually dropped from 52.2% success ( $SD = 34.3\%$ ) at pretest to 50.6% success ( $SD = 28.4\%$ ) at posttest, while performance in the experimental group improved from 45.9% success ( $SD = 36.1\%$ ) at pretest to 56.8% success ( $SD = 31.3\%$ ) at posttest. It thus appears that the experimental instruction positively influenced students' problem-solving performance. We next consider the specific nature of these improvements and how they compare to improvements made by the control group (see figure 2).

Gains for students in the control classes were almost exclusively limited to linear problems in which the mathematical representation presented was a symbolic equation. This is perhaps not surprising given the control classrooms' use of the *Connected Mathematics* seventh grade curriculum, which focuses largely on linear relationships



represented symbolically and graphically. However, the gains in symbolic-linear problems were offset by losses in word-linear problems.

On the other hand, student gains in the experimental classes that used Bridging Instruction were much broader, with improvement dispersed across both levels of linearity and both verbal and symbolic problem representations. This is consistent with the attention the experimental instruction gave to exposing students to multiple mathematical representations and patterns of change. These gains also make clear that seventh and eighth grade students can in fact improve their reasoning about nonlinear functions in various representational formats.

As noted, the control curriculum did not address nonlinear relationships during the time in which the study took place, whereas the experimental curriculum addressed both linear and non-linear relationships. As both curricula were implemented over the same 9-week period, it is clear that the experimental curriculum devoted less time to linear functions relative to the control curriculum. However, despite the control curriculum's exclusive emphasis on linear situations and the experimental curriculum's more dispersed focus, gains on linear problems were actually greater for the experimental group. The control group dropped from 62.0% correct (SD = 31.2%) at pretest to 61.3% correct (SD = 22.2%) at posttest, for a percentage loss of 0.7% (those losses occurring on graph and word expression problems), while the experimental group improved from 54.7% correct (SD = 33.2%) at pretest to 64.8% correct (SD = 26.4%) at posttest, for a percentage gain of 10.1%. Within the same time period students using BI also showed a marked improvement solving problems with nonlinear relations (pretest  $M = 37.0%$ ,  $SD = 37.3%$  and posttest  $M = 48.8%$ ,  $SD = 34.2%$ ).

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Place Figure 2 about here

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Finally, item analyses of problem-solving gains also uncovered a significant three-way interaction among problem representation, linearity, and date,  $F(2, 12) = 4.62, p = 0.02$ . Given a graphical representation, student performance held fairly steady across date and linearity (most likely due to ceiling effects). Given a word expression representation, students showed greater pre-to-posttest gains on nonlinear problems than linear problems. Given a symbolic representation, students showed greater pre-to-posttest gains on linear problems than on non-linear problems, which remained largely unchanged.

### *Translation*

Recall that translation problems are ones in which the student is asked to produce a representational “output” (a graph, symbolic equation, table, or word expression) that contains information mathematically equivalent to the representational “input” (a graph, symbolic equation, or word expression) presented in the problem. Example translation items are shown in Appendix A.

*Translation pretest.* Overall translation pretest performance was quite poor, with slight success only with a small number of input-output pairs. Overall mean scores were 11.2% correct (SD = 20.7%) for the control group and 8.3% correct (SD = 20.4%) for the experimental group. Based on the pretest data in Figure 3, one can see that success varied across the input-output pairs. Student performance differed significantly from zero only on graph-to-table and word expression-to-symbol translations, with performance on word-to-table translations being marginally better than zero.

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Place Figure 3 about here

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Students performed better on translation items that involved linear relations than on items that involved non-linear relations,  $F(1, 68) = 12.36, p = 0.0008$  across participants and  $F(1, 4) = 21.91, p = 0.009$  across items.

*Translation gains.* Considering the pretest and posttest data together, we again found that linearity had a significant main effect on students' translation performance,  $F(1, 68) = 23.27$ ,  $p < 0.0001$  across participants and  $F(1, 4) = 30.63$ ,  $p = 0.005$  across items. Slope-sign did not significantly impact translation performance, as was found with problem-solving items.

Overall, students were more successful on the translation items at posttest than at pretest,  $F(1, 68) = 42.86$ ,  $p < 0.0001$  for participant and  $F(1, 4) = 14.89$ ,  $p = 0.02$  for item. However, students in the BI group made greater gains than students in the control group, as revealed in tests of the condition by date interaction, which was significant in the participant analysis,  $F(1, 68) = 5.94$ ,  $p = 0.02$ , and which approached significance in the item analysis,  $F(1, 4) = 5.76$ ,  $p = 0.07$ . The control group experienced a 5.6% increase in correct responses, while the experimental group experienced a 17.6% increase. Gains for each input-output combination are presented in Figure 4.

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Place Figure 4 about here

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Although students' translation performance did improve from pretest to posttest, overall posttest scores were still quite low, as seen in Figure 5. As at pretest, the given input-output pair again influenced students' translation success at posttest. Students in the CM group learned to translate from any of the input representations to a table of values. Students in the BI group made even broader improvements, learning to translate from any of the input representations to tables of values and to graphs, as well as between word expressions and symbols. This finding suggests an emerging reciprocity between words and symbols that may be based on their mutual compatibility as holistic and propositional representations.

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Place Figure 5 about here

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CM students' relatively good performance on table-output translations and BI students' relatively good performance on table-output and graph-output translations invites us to consider the arithmetic nature of these tasks. To succeed on each of these tasks, students needed to generate legitimate instances from the given function. For graphs, this required reading and recording specific points (instances) in the function along the plotted contour. In contrast, to succeed with word and symbolic equations students needed to substitute values into the given relation and record the corresponding value. The relatively large performance gains on instance-based input-output relations are in stark contrast to the small gains observed with more holistic verbal expressions and symbolic equations.

This contrast between success on instance-based versus holistic tasks can be highlighted not only across representations but—in the case of graphs—within representations. Recall that in order to qualify as correct, graphs only needed to have three correct data points and no incorrect ones. It was not necessary for the axes to be labeled with words, for the points to be connected with a line or exponential curve, or for the y-intercept to be included. Had these additional requirements been in place, the graph production task would have required a more holistic understanding of the representation and the function it described.

We recoded the graph-output translation data using these more “strict” criteria, to evaluate whether students also possessed a more holistic understanding of the function described. With these new holistic criteria in place, performance across dates and conditions was very poor, with both groups having 0% success at pretest, and neither group significantly above zero at posttest (Control  $M = 4.2\%$ ,  $SD = 8.3\%$ ; Experimental  $M = 4.3\%$ ,  $SD = 8.1\%$ ). Clearly, a holistic understanding of graphical representations was far

less attainable than an instance-based one. The comparative demands of point-wise and holistic representations are addressed further in the Discussion section.

### Discussion

We now reconsider the empirical findings in light of our original research questions and discuss the broader implications of representation use in terms of students' quantitative problem solving and learning.

#### *How Problem-Solving Performance is Influenced by Representation: The Representation-Complexity Interaction*

The first research question explored how problem-solving performance is influenced by representation. The findings suggest that students' initial problem-solving performance is heavily influenced by the specific characteristics of the various representational formats that are commonly found in algebraic activities. Chief among these findings is that problem-solving performance using pre-constructed graphical representations exceeds that of all of the other matched representations. Performance success with graph-based problems averaged 85.7%, fully 46.7% ahead of word expressions, the closest contender, and 63.3% ahead of symbols. This graphical advantage holds regardless of slope-sign (i.e., increasing or decreasing) for both linear and nonlinear functions.

Algebra instruction in classrooms and in textbooks emphasizes use of symbolic representations (Mayer, Sims & Tajika, 1995). Students' level of success with pre-constructed graphs suggests that graphs could play a more central role in the development of early algebraic reasoning than they do in current curricula. In the language of *Bridging Instruction*, graphs may serve as a conceptual grounding for new concepts and procedures, just as verbal representations have in previous investigations (Koedinger & Nathan, 1999; Nathan & Koedinger, 2000c).

The second finding that warrants attention concerns the representations that best serve linear and nonlinear problem solving. Pretest data showed greater success when the problems were presented in a verbal format for linear problems than when they were symbolically presented. However, when problems dealt with nonlinear relations, symbols proved to be more effective than verbal representations. As noted, this result replicates earlier findings showing a complexity-based trade-off among representations. Previous work (Koedinger, et al., 2001) has shown that simple algebra problems involving a single occurrence of an unknown variable are solved most readily when the problems are presented in words. However, for high complexity algebra problems that used multiple occurrences of the unknown (e.g.,  $x - 0.15x = 38.24$ ), students performed better when they problems were presented symbolically. This interaction between complexity and representation was due in part to the fact that verbal problem formats tend to elicit highly reliable arithmetic-based solution strategies (such as guess-and-test and working backwards) , which work well when solving simple relations. However, with increasing complexity the verbal formats quickly become computationally unwieldy, while symbol-based representations “scale up” far better. This representational trade-off was apparent among remedial college students ( $n = 153$ ), and college students with extremely strong mathematical reasoning skills (mean SAT = 719 out of 800;  $n = 65$ ).

In a parallel fashion, the current study showed (see Figure 1) that verbal representations are most effective when solving the lower complexity linear problems (linear word pretest  $M = 65.1\%$ ,  $SD = 22.3\%$ ; linear symbol pretest  $M = 24.0\%$ ,  $SD = 21.0\%$ ), whereas symbolic representations are more effective than words for higher-complexity non-linear problems (nonlinear word pretest  $M = 13.0\%$ ,  $SD = 12.3\%$ ; nonlinear symbol pretest  $M = 20.8\%$ ,  $SD = 19.8\%$ ).

In addition to replicating the basic representation-complexity interaction, the current study extends it by including graphical representations. As is evident in Figure 1, graphs “scale” nicely with the shift in complexity. This makes sense as reading a point off the

graph is not much more difficult for the nonlinear graphs used in the assessment than for the linear graphs. It also underscores the high utility of graphical representations across a range of relations.

*How Representation Use is Enhanced by Instruction: Bridging Instruction and Contrasting Cases*

Our second research question explored the effect of a 9-week instructional intervention on students' use of various representations for algebraic problem solving. Instruction in both the nationally marketed *Connected Mathematics* program (Lappan et al., 1998a, 1998b) and the newly developed Bridging Instruction approach (Koedinger & Nathan, 1999; Nathan & Koedinger, 2000c) led to measurable gains in students' algebra problem solving. Gains were small for graph-based problems, most likely because students had little room for improvement on average, given their strong pretest performance (Figure 1). Gains with symbolically presented problems were the largest, and showed that both instructional approaches improved students' abilities to reason about equations.

However, some important differences emerged between the two conditions. Students in the classes that used the BI approach showed larger gains than the students in the CM classes, even though all were taught by the same teacher. CM students, who received instruction exclusively on linear relations, only showed gains in linear, symbolic problem solving (from 28.9% to 46.8%, a 17.9 point gain). In contrast, BI students, who received instruction in both linear and non-linear relations during the same time period, showed comparable gains in linear, symbolic problem solving (from 19.0% to 39.2%, a 20 point gain), superior gains in linear problem-solving overall (collapsing across representational formats, +10.1%), and significant gains in non-linear problem solving (collapsing across representational formats, +11.8%). The data also showed that whereas improvement in the CM group was limited to symbolic representations, students in the BI condition showed gains in symbol use as well as significant gains in word expressions,

particularly for non-linear problems (nonlinear CM word gain = -1.5%, nonlinear BI word gain = 22.8%).

One interpretation of these findings is that the BI approach engendered more learning than CM in the same amount of time – an extremely positive result for a theoretically based curriculum unit. We present some plausible hypotheses for why BI may have produced larger and broader gains in problem solving performance. First, it is possible that teaching students about linear and non-linear functions is beneficial to their learning of each. Each type of function may actually serve as a contrasting case (Schwartz & Bransford, 1998) for the other, reinforcing important common concepts while helping students attend to the important differences between them. It is also plausible that by explicitly connecting algebraic representations and procedures to students' invented strategies and representations, the Bridging Instruction approach provides a conceptual grounding for the meaning of these representations that supports fluency. This would corroborate and extend prior studies of early algebra learning that built on students' invented strategies and representations for algebra story problem solving (e.g., Nathan & Koedinger, 2000c). The present study does not allow us to make any strong claims about the individual and combined effects of bridging or of teaching linear and non-linear functions in tandem. Further studies of the contributions of each of these toward the development of students' representational fluency are needed to be more conclusive.

#### *How Well Early Algebra Students Translate Among Representations:*

##### *Differential Demands Between Comprehension and Production*

The third question that guided this research focused on students' pre-intervention fluency among graphical, tabular, verbal and symbolic representations. The data on translation tasks makes it clear that representational fluency is an advanced skill: students struggled to move among tabular, graphical, verbal and symbolic forms of representation far more than they struggled to use individual representational formats to solve problems.



At pretest, performance was essentially at zero for all but a few types of translation problems. Students could translate from verbal rules to symbolic equations (performing at about 30%), and could map from graphs to tables (at about 20%). Students are better at linear than non-linear relations, but there is tremendous room for improvement in all areas of translation that we explored.

These data point to an important difference in the psychological demands between *comprehending* a representation and *producing* one. Of particular interest here is the differential performance in graph use between translation and problem solving. Students could use pre-constructed graphs at about the 80% level for problem solving. However, their performance levels dropped to about the 12.6% level when they had to produce them. Pre-intervention graph use is generally poor among US students, owing, in part, to its relative absence in standard curricula (e.g., Demana, Schoen & Waits, 1993). Tasks that use pre-constructed representations that serve as effective memory cues allow students to access knowledge that would otherwise remain forgotten (cf. Tulving & Thompson, 1973). Yet the differing demands for problem solving and production-intensive tasks can be overlooked by practitioners. This important distinction must be acknowledged whenever students' representational fluency is being assessed.

#### *How fluency among representations is enhanced by instruction*

The final question that guided this research had to do with the effects of instruction on students' representational fluency. At posttest, students in both groups succeeded on problems that required them to produce tables of values from words, graphs or equations. Students in the BI group, in addition to those gains, also succeeded on problems that required them to produce graphs from word expressions and symbolic equations. Gains on symbol and word output tasks were relatively minimal. The patterns of success that contrast high table and graph output with low symbol and word output suggest that attributes of the representations themselves are important factors in students' learning.

One interpretation of these data is that students gained most when they could generate representations that were *instance-based* (i.e., point-wise) or *local* in their representational scope. Tables of values naturally have this instance-based emphasis, as reflected by the scoring criteria. Consequently, successful performance can be achieved by successfully specifying independent points through predominantly arithmetic means. In contrast, students struggled with more *holistic* representations, such as word and symbolic equations. Holistic representations like equations require students to abstract from specific instances and capture the covariation of a function for all values (in this case, the infinitely large set of all real numbers) in a concise, unified, and syntactically accurate description.

Graphs present an interesting case within the instance-based/holistic dimension as they can take on a local flavor (as with scatter plots and bar graphs) or can be holistic (as with line graphs). To further understand students' performance with representations along this dimension, we compared experimental students' post-intervention abilities to produce proper graphs when they were judged with instance-based and holistic scoring criteria. When experimental students' graphs were evaluated from a more lenient, point-wise perspective, students exhibited relatively high levels of performance ( $M = 29.5\%$ ). When they were evaluated with the more stringent, holistic criteria, performance was much lower ( $M = 4.3\%$ ).

Our findings are consistent with Kalchman, Moss, and Case's (2001; Kalchman, 1998) theory of developmental progression for children's understanding of mathematical functions. Kalchman's model holds that, in the minds of students, procedurally based (i.e., computational) representations, such as tables of instances, and analogical representations of mathematical functions, such as bar graphs, developmentally precede and form the basis for the more integrative, geometric representations, such as line graphs. From this, one would expect to see greater success with instance-based graph and table output problems as shown in the current work. Our work also shows that graphs can be used in either an instance-based or holistic fashion, which follows from its twofold nature as an integrative

representation. This duality is not as apparent with symbolic, word, or table representations.

A plausible hypothesis is that tables and instance-based graphs may naturally serve as entry points into the mathematics of covariation, which serves as a central idea for the mathematics of functions. Graphs may be particularly effective in helping students to bridge to more holistic representations. Ultimately, they may help students to learn the symbolic formalisms and reap their rewards in the face of increasingly complex relationships.

### *Conclusion*

In sum, the present study investigated early algebra students' abilities to work within and translate among various representational formats. On the whole, students were more successful at solving problems than at translating among representations. However, performance was strongly dependent on the particular representational formats used. Student performance improved with instruction, and the greatest gains were found for an experimental curriculum (*Bridging Instruction*) that focused on both linear and nonlinear relations, and the made explicit links to students' invented strategies and representations. Students' performance suggests that there are often significant gaps between their abilities to comprehend particular representations and their abilities to produce those representations. Finally, students appear to attain fluency with instance-based representations (such as tables and point-wise graphs) before they attain fluency with more global, holistic representations (such as symbolic equations and verbal expressions). These findings highlight the challenges of fostering representational fluency in early algebra instruction, and indicate areas for future curricular development.

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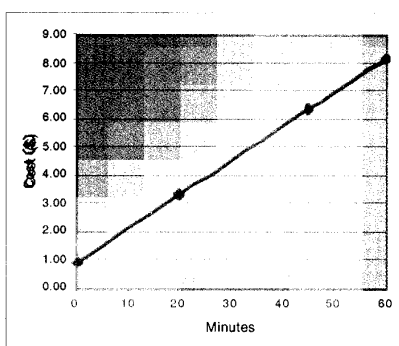
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## Appendix A

Table A1. Example of multiple forms of a linear problem.

Problem Section	Information Presented
Situation introduced	Cassandra sells phone cards to college students so they can make long distance calls for a good price. Each card has a base charge and a per-minute rate.
Input presented	<i>Graph Input:</i> Below is a graph you could use to find the price of the card if you know the number of minutes on it.



*Symbol Input:* The expression below shows how to find the price of the card,  $p$ , of you know the number of minutes on it,  $n$ .

$$p = 0.99 + 0.12n$$

*Word Expression Input:* The description below tells you how to find the price of the card if you know the number of minutes on it.

To find the price of the card, you multiply the number of minutes by the per-minute rate of \$0.12, and then add the base charge of \$0.99.

---

Part a (problem solving) What would be the price of a card with 30 minutes?

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Part b (problem solving) How many minutes would be on a card that cost \$6.99?

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Part c (translation)

*Graph Output:* Make a graph that you could use to find the price of the card if you know the number of minutes. (*Examination packets had graph paper included.*)

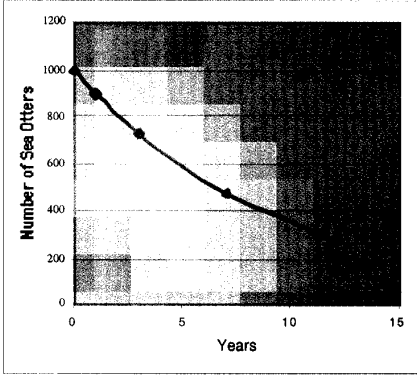
*Symbol Output:* Write a mathematical expression that tells how to find the price of the card if you know the number of minutes.

*Table Output:* Make a table of values that you could use to find the price of the card if you know the number of minutes.

*Word Expression Output:* Describe in words how to find the price of the card if you know the number of minutes.

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Table A2. Example of multiple forms of a nonlinear problem.

Problem Section	Information Presented
Situation introduced	Marine biologists are concerned that the population of sea otters is rapidly decreasing in one area.
Input presented	<p><i>Graph Input:</i> Below is a graph that you could use to find the population of sea otters in that area, if you know the number of years since the study began.</p> 
	<p><i>Symbol Input:</i> The expression below shows how to find the population of sea otters in that area, <math>p</math>, if you know the number of years since the study began, <math>n</math>. <math>p = 1000(1 - .10)^n</math></p> <p><i>Word Expression Input:</i> To find the population, you take 1 minus 0.10 and raise it to the power of the number of years since the study began, and then you take the result and multiply it by the starting population of 1000.</p>
Part a (problem solving)	What was the population of sea otters after 5 years of the study?
Part b (problem solving)	What was the population of sea otters when the study began?

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Part c (translation)

*Graph Output:* Make a graph that you could use to find the population of sea otters if you know the number of years since the study began.

*(Examination packets had graph paper included.)*

*Symbol Output:* Write a mathematical expression that tells how to find the population of sea otters if you know the number of years since the study began.

*Table Output:* Make a table of values that you could use to find the population of sea otters if you know the number of years since the study began.

*Word Expression Output:* Describe in words how to find the population of sea otters if you know the number of years since the study began.

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## Figure Captions

*Figure 1.* Proportion of problem-solving pretest items solved correctly by problem representation and linearity.

*Figure 2.* Control and experimental group gains on symbol and word expression representations by linearity.

*Figure 3.* Students' pretest success producing graph, symbol, table, and word expression outputs given graph, symbol, and word expression inputs. Asterisks (\*) indicate means significantly different from zero,  $p < .05$ .

*Figure 4.* Student gains producing graph, symbol, table, and word expression inputs given graph, symbol, and word expression inputs.

*Figure 5.* Students' posttest success producing graph, symbol, table, and word expression outputs given graph, symbol, and word expression inputs. Asterisks (\*) indicate means significantly different from zero,  $p < .05$ . For Word input to Symbol output for the control group,  $p < .10$ .

Table 1. Comparison of the experimental and control curricula.

Assessment Dimensions	BI Curricula	CM Curricula
<b>Use of Representations</b>		
Words	<All>	V, M
Tables	B&P, PC, CP, MT	V, M
Graphs	B&P, BS, CP, MT	V, M
Symbols	BS, PC, PF, CP, MT	V, M
Translation between representations	<All>	V, M
Linear	B&P, PC, CP, MT	V, M
Non-linear <sup>†</sup>	BS, PC, PF, CP, MT	--
Increasing functions	B&P, BS, PF, CP, MT	V, M
Decreasing functions	PC, MT	M**

B&P = Bridges and Pennies; BS = Building Squares; PC = Paper Cutting; PF = Paper Folding; CP = Cube Problem; MT = Matching Task.

Vi = “Variables and Patterns” (CM) unit i

Mi = “Moving Straight Ahead” (CM) unit i

\* CMP Investigation 5: “Exploring Slope”

\*\* Note: The “Bridges and Pennies” activity was adopted from the CM curriculum

† quadratic, cubic, logarithmic relations.

