Connecting Research to Teaching

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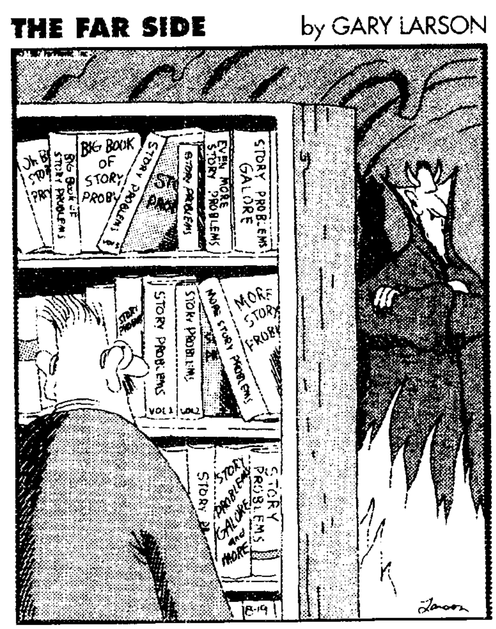
Moving beyond Teachers' Intuitive Beliefs about Algebra Learning

Think about this question: Why are algebra story problems considered to be the most difficult tasks facing algebra students? Teachers generally regard them as difficult (Nathan and Koedinger forthcoming), textbooks typically place these problems at the ends of chapters (Nathan and Long 1999), students find them least favorable, and even comic-strip folklore presents story problems as the bane of formal education. (See **fig. 1**). Are these perceptions held by teachers and textbook authors justified? This article examines teachers' judgments of the difficulty of algebra problems. Our findings may surprise readers, and we hope that they will motivate readers to reexamine some long-standing assumptions about mathematics learning and instruction.



Hell's library

Fig. 1
Story problems are commonly presented as the most undesirable of tasks.

Beliefs that teachers hold about students' abilities most influence instructional practices

One thing seems certain. However mathematical difficulties are portrayed, the beliefs that teachers hold about students' mathematical abilities and learning processes most influence teachers' pedagogical decisions, planning activities, and instructional practices (Borko and Shavelson 1990). Teachers cite students' ability as the characteristic that has the single greatest impact on instructional decisions. Because teachers' views of their students are so important, we investigated the relationship between teachers' perceptions and students' actual performances on a set of mathematics tasks. Our goals for this article are to give an accurate picture of students' abilities and to dispel some myths about student performance at the algebra level. Furnishing a realistic picture of students' mathematical development is especially important because schools throughout the United States plan to offer algebraic instruction for all students and at earlier grades. At the end of this article, we offer classroom-tested instructional approaches that build directly on students' mathematical intuitions and inventions.

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THE PROBLEM-DIFFICULTY RANKING TASK

We invite readers to think about their expectations about students' performance on the small set of problems presented in the Problem-Difficulty Ranking Task in **figure 2.** A similar version of the task was given to groups of mathematics teachers attending district-sponsored professional development workshops (Nathan and Koedinger forthcoming).

Before reading our report of the views commonly held by mathematics teachers, readers should spend a few minutes performing the ranking task shown in **figure 2.**

TEACHERS' PREDICTIONS

Table 1 organizes the problems in the ranking task along two dimensions. The rows of the table show problems that are either arithmetic, that is, the result is unknown, or algebraic, that is, a starting quantity is unknown. The columns indicate a verbal or a symbolic format. A word-equation problem, such as problem 2, is similar to the pick-a-number game (Usiskin 1997). Unlike a story problem, a word equation verbally describes the relationship among pure quantities with no story context.

When high school teachers judged the relative difficulty of the problems shown in **figure 2,** they generally considered the arithmetic problems—problems 4, 5, and 6—easier than the algebra problems—problems 1, 2, and 3—regardless of their presentation in word, story, or symbolic format (see **table 2**). Teachers also thought that the verbal problems—word and story problems like problems 1–5—were more difficult for students than the symbolic problems. Finally, the algebra word and story problems were ranked most difficult for students.

Readers should compare their own predictions with our findings. Although some variability certainly exists in the responses, we have found the same pattern with several groups of teachers, including high school teachers, around the United States, and with mathematics educational researchers who focus on algebra learning and instruction (Nathan and Koedinger forthcoming; Nathan, Koedinger, and Tabachneck 1996). We next compare teachers' predictions with student performance.

A word equation describes the relationship among pure quantities with no story context

STUDENTS' PERFORMANCE AND TEACHERS' EXPECTATIONS

We examined the problem-solving performance of 171 high school students near the end of their first-year-algebra course. These urban students took a

TABLE 1

The Underlying Structure of the Problems Given in the Difficulty-Ranking Task

Area of Mathematics	Presentation Type: Verbal — Story	Verbal — Word Equation	Symbolic — Symbol Equation
Arithmetic	P4. When Ted got home from his waiter job, he took the \$81.90 that he earned that day and subtracted the \$66 that he received in tips. Then he divided the remaining money by the six hours that he worked and found his hourly wage. How much does Ted make per hour?	P5. Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?	P6. Solve for x: $(81.90 - 66)/6 = x$
Algebra	P1. When Ted got home from his waiter job, he multiplied his hourly wage by the six hours that he worked that day. Then he added the \$66 that he made in tips and found that he earned \$81.90. How much does Ted make per hour?	P2. Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?	P3. solve for x: $x * 6 + 66 = 81.90$

THE PROBLEM-DIFFICULTY-RANKING TASK

Below are six problems that are representative of a broader set of problems that are typically given to public school students during middle school and high school mathematics courses.

What we would like you to do

Take about five minutes to complete this task. Rank order the problems below on the basis of your expectations of their difficulty for students in your class, starting with the ones that you think will be easiest for these students and progressing to the ones that you think will be harder. You can have ties if you like. For example, if you think that the fourth problem was the easiest, the third was the most difficult, and the rest were about the same, you would write

4 (easiest)
2 1 5 6
3 (hardest).

Problems

1) When Ted got home from his waiter job, he multiplied his hourly wage by the six hours that he worked that day. Then he added the \$66 he made in tips and found that he earned \$81.90. How much does Ted make per hour?
2) Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?
3) Solve for x: $x \cdot 6 + 66 = 81.90$.
4) When Ted got home from his waiter job, he took the \$81.90 that he earned that day and subtracted the \$66 that he received in tips. Then he divided the remaining money by the 6 hours that he worked and found his hourly wage. How much does Ted make per hour?
5) Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?
6) Solve for x: $(81.90 - 66)/6 = x$.

Fig 2
Problem-difficulty ranking task similar to that given to teachers

TABLE 2
Teachers' Rankings, Students' Performance, and Textbook Order of Problem Types

	Teachers Rank Ordering (n = 67)	Student Performance and Percent Correct (n = 171)	Textbook Order
Easy	(P6) Arithmetic—eqn (P5) Arithmetic—word (P4) Arithmetic—story	(P4) Arithmetic—story (80%) (P5) Arithmetic—word (74%)	(P6) Arithmetic—eqn (P5) Arithmetic—word (P4) Arithmetic—story
Medium	(P3) Algebra—eqn	(P1) Algebra—story (60%) (P6) Arithmetic—eqn (56%) (P2) Algebra—word (48%)	(P3) Algebra—eqn
Hard	(P1) Algebra—story (P2) Algebra—word	(P3) Algebra—eqn (29%)	(P2) Algebra—word (P1) Algebra—story

test consisting of problems similar to those in the ranking task (Koedinger and MacLaren 1997). As teachers predicted **(table 2),** students found that arithmetic problems were easier to solve than algebra problems. They correctly solved 70 percent of the arithmetic problems, whereas they correctly solved 46 percent of the algebra problems.

Students' verbal and symbolic problem-solving performance contradicted teachers' expectations, however. Students correctly solved approximately 65 percent of the word and story problems but only 43 percent of the symbolic-arithmetic and algebra problems. Students scored well on story and word-equation problems regardless of the presence of a problem context. Surprisingly, the accuracy rate for arithmetic equation problems, 56 percent, was nearly the same as for algebra story problems, 60 percent, suggesting that introductory algebra story problems may not deserve their infamous reputation.

Teachers are not surprised that arithmetic problems are easier than matched algebra problems, but many have difficulty believing that students are more successful on story problems than on comparable equations. Column 1 of **table 2** shows the relative difficulty ranking given by teachers, and column 2 shows the order of problem-solving difficulty for students. The difficulty divisions shown—easy, medium, and hard—reflect statistically significant differences ($p < .05$).

Overall, teachers predicted much of what makes problems difficult for students, and they are more often right than not. However, the comparison also shows discrepancies that may be based on the intuitive beliefs that teachers have about students' mathematical reasoning. Most notably, teachers expect story and word problems to be most difficult, whereas students actually find that symbolic equations are hardest. This preconception is pronounced and systematic, and it may significantly influence teachers' curricular decisions.

Problem. Katrina's allowance is \$2.75 more than Sarah's allowance. The combined allowances of the two girls total \$14.75 each week. What are their allowances?

Student Written Work	Student Speech
Step 1 K - S = \$2.75 K + S = \$14.75	S: Katrina, K, is two seventy-five more than Sarah, S. K and S is fourteen dollars and seventy-five.
Step 2 K = 7 S = 7.75 7 - 7.75	S: So, seven and seven point seven five is fourteen point seven five. But . . . too small [referring to the difference].
Step 3 6 - 8.75 8.75 - 6 = 2.75	S: Go up here [writes 6] . . . and down one here [writes 8.75]. . . . Oh! Switch them, so eight point seven five minus six . . . is . . . . Yea!

Fig. 3
The guess-and-test strategy

A closer look at students' solution strategies

Readers may wonder how story problems can possibly be easier than equations. After all, many teachers sensibly argue, the way to solve a story problem is by translating it to an equation and then solving the equation. Alternatively, teachers have commented informally that solving equation problems should be easiest because they present "pure mathematics," whereas solving story problems requires "applying" the mathematics. This view is also presented in most mathematics textbooks, as well as in the research literature (e.g., Mayer [1985]). It is a simple logical conclusion, then, that story problems must be harder, since equation solving is just one of two steps for solving a story problem.

When we examine how students solve verbal mathematics problems, we see that their strategies explain why story problems are easier for them to solve. Analyses of beginning-algebra students' solution methods reveal that they do not typically follow the two-step translate-and-solve approach. Students frequently use such informal approaches as guess-and-test and unwinding to solve algebra word problems when they are allowed to choose a solution method (Hall et al. 1989; Kieran 1992; Koedinger and MacLaren 1997, Nathan and Koedinger forthcoming). The guess-and-test approach uses arithmetic procedures in a forward manner to solve algebra word problems iteratively after substituting a value for the unknown quantity (see **fig. 3**). In the unwinding method, students work backward through the quantitative relations of an algebra problem by inverting the mathematical operations and the order of quantities, as shown in **figure 4.** Unwinding circumvents the need for symbol manipulation by transforming the algebra story problem into a sequence of arithmetic tasks.

With alternative solution approaches, students use their knowledge about the world to facilitate their reasoning. For example, by keeping the numbers in a monetary context in the unwinding solution shown in **figure 4,** the student avoids misaligning the place values—a common error during subtraction. The informal methods that we have seen are successful about 70 percent of the time that students apply them to algebra story problems, whereas the formal methods produce correct solutions about 30 percent of the time. Students' use of informal methods increases their problem-solving accuracy on the problems that teachers expect to be most difficult.

WHERE DO TEACHERS' VIEWS ORIGINATE?

If the views expressed by teachers in the ranking task are so widely held, where do they originate? And if teachers' beliefs are not supported by data, why do they persist? One answer is that they fit into a view of learning that is deeply rooted historically and promoted implicitly by many mathematics textbooks (Greeno, Collins, and Resnick 1996). Algebra and prealgebra textbooks are often organized from a symbol-precedence view. They present symbolic problems early in arithmetic and algebraic lessons, whereas verbal problems typically appear at the end, as "challenge" or "application" problems (Nathan and Long 1999). The idea that instruction should build on what students already know is a basic principle of learning. Teaching symbolic problems before teaching story problems makes sense under the assumption that symbolic problems are easier. However, our studies indicate that this assumption is incorrect.

Although teachers' expectations parallel the problem sequence found in many textbooks, students' problem-solving performances do not (**table 2**). Our work (Koedinger and MacLaren 1997; Nathan and Koedinger forthcoming) shows that the performance of only 46 percent of the high school students studied is consistent with the textbook view. In contrast, the performance of 88 percent of students is consistent with the verbal-precedence view of algebra development that indicates that verbal problem-solving skills develop before symbolic reasoning does.

BRIDGING FROM STUDENTS' IDEAS TO FORMAL ALGEBRA INSTRUCTION

Our work shows that most students do not acquire symbol-manipulation skills at the levels that high school teachers expect. It suggests that teachers need to look for new ways to teach algebra concepts and skills that build directly on students' informal solution methods and their verbal skills. We next discuss some implications of our work for teaching algebra at the middle school and high school levels and present a few specific instructional principles and approaches that we have used in the classroom

Problem. Minerva bought three pairs of the jeans that she loves. In the next store, she bought a pair of shoes for $64.00. She spent a total of $125.50. How much did each pair of jeans cost?

Student Written Work	Student Speech
Step 1 (with error): Undo the price of shoes. $125.50 - 64 $124.86	S: One hundred and twenty-five dollars and fifty cents, minus, MINUS sixty-four dollars for the shoes. So she has a remainder of one hundred and twenty-four dollars and eighty-six cents
Step 1 (without error): Undo the price of shoes. $ 125.50 - $ 64.00 $ 61.50	S: Oh! [S catches her error and performs a new subtraction.] She has a remainder of sixty-one dollars and fifty cents.
Step 2: Undo the price of each pair of jeans. $ 20.50 3) 61.50 - 6 01 - 00 15 - 15 00	S: Now, divide that by three. They are each twenty bucks. Twenty dollars and fifty cents. [Uses calculator to check the division calculation.]

Fig. 4
The unwinding strategy

to extend students' algebraic reasoning.

Students' invented methods for solving algebra-level problems are diverse and can lead to very different patterns of performance than might be expected from commonly held views of mathematical development. Most students develop capabilities for reasoning verbally about quantities before developing symbolic-reasoning skills. Students' early ideas about algebraic relations supply a firm footing on which to build formal symbolic methods.

One of our general principles of instruction is to identify students' intuitive solution methods so that we can build on their prior mathematical thinking. Students already have in place much of the conceptual foundation for algebraic reasoning. However, students and teachers are usually unaware of the power and legitimacy of students' intuitions and invented solution methods. Thus, our second principle for instruction is to make students' thinking visible to teachers and to the students themselves. We offer examples of algebra instruction that demonstrate how these principles can guide teaching.

Students as young as the sixth grade can discuss their approaches in depth with their peers (Nathan, Knuth, and Elliott 1998). This activity, in turn, supports students' ownership of mathematical ideas. However, the teacher must select exercises carefully. If problems are too easy, that is, if they use very common number facts, students may apply their methods automatically, without really noticing the solution approach that they used. Choosing numbers that require students to write things down and

Verbal problem-solving skills develop before symbolic reasoning

present all their steps helps them see their own reasoning processes. Teachers may also introduce the notion of a "problem solvers' toolkit," that is, a collection of solution methods that students can list in their notebooks or on a classroom bulletin board. Students can then discuss their strategy decisions explicitly during problem-solving activities and can reflect on the factors that influenced their choices. Explicit discussion of the trade-offs of strategies, for example, efficiency and computational difficulty, raises the level of classroom discourse by focusing on solution methods and problem demands rather than on computations and answers.

Bridging from the guess-and-test strategy

Students' informal solution strategies can serve as excellent conceptual bridges to formal symbol-manipulation procedures. For example, the guess-and-test method highlights the structural aspects of algebraic equations (Kieran 1992). The iterative process relies heavily on students' number facts and encourages students to explore how relations among the quantities act as constraints. It also grounds the concept of variable for the student. In classes, we have seen students use the guess-and-test method on problems with multiple constraints and unknowns, such as the following problem:

Informal solution strategies can serve as bridges to formal symbol-manipulation procedures

(*a*) ◇ + ⬡ = ?

(*b*) ◇ − ⬡ − ⬡ = ?

First, students listed all that they knew about (*a*) and (*b*) when the right-hand side of the equation was unspecified. Students speculated that the diamond and the hexagon could be worth anything in (*a*) but that the value of the diamond in (*b*) had to be greater than the value of the hexagon if the values were to be positive. Students agreed on the restriction to positive values because they had had little experience with negative numbers. Students speculated that the shapes had to have the same values across the two equations. Others then noted that the diamond had to be at least twice the value of the hexagon. The teacher listed all the things that students agreed that they "absolutely knew" from (*a*) and (*b*) as given.

The teacher then said that equation (*a*) equaled 45 and equation (*b*) equaled 24. "Now," the teacher asked, "what do we know?" Students noticed that the values of the two shapes had to be between 0 and 45. They said that the sum of the two shapes had to be 45—no more, no less. Students started guessing values and testing them, using their previous record of information. Some guesses worked for one equation but not for both. Students used information from initial guesses to revise later ones. They made lists of guesses, for example, all the pairs of numbers that sum to 45. Over time, it became tedious to draw the shapes, so students began using such shortcuts as the following:

(*c*) $D + H = 45$

(*d*) $D - H - H = 24$

Finally, they found an answer that worked, that is, $D = 38, H = 7$.

The teacher next led students in a discussion of their strategies. The shortcuts—equations (*c*) and (*d*)—became "algebraic equations." The shapes-turned-letters became "variables" because these values were the ones that students kept varying. The values of the variables had to be "consistent." The equations had to "balance" to avoid contradictions. In our view, students were bridging from their intuitive conceptualizations to formal representations of a problem, so we decided to call this approach *bridging instruction* (Koedinger and Alibali 1999; Nathan 1999). This instructional technique is a powerful way to make symbolic representations meaningful to students.

Bridging from the unwinding strategy

The unwinding method is another informal strategy that stems from students' intuitions about manipulating quantities. Even young students naturally "undo" the relations presented in a story problem, as shown in **figure 4**, and use separate arithmetic calculations to carry out the unwinding procedure. Unwinding provides a natural introduction to inverse operations and equation balancing—two crucial aspects of symbolic-skill development. However, we ultimately want students to be able to advance beyond unwinding and develop formal reasoning skills for symbolic representations. Again, we have found that bridging from students' invented strategies to formal methods is an effective instructional approach. One useful device to make the bridge from unwinding to formal algebraic reasoning is the solution summary. The solution summary captures the separate arithmetic steps used by a student who is calculating an answer, such as the cost of each pair of jeans in **figure 4;** compare **figure 4** with equation (*e*). By being introduced in several contexts, the solution summary becomes assimilated into the classroom discourse.

After the solution summary became a familiar idea in the classroom, the teacher introduced the situation equation, equation (*f*). We needed two class periods with sixth graders. As experts, we see that the situation equation models the quantitative relations of the story situation in the original problem. However, students generally attend more directly to the solution method than to the underlying mathematical model of the situation. Instead of introducing the situation equation as a translation of the relations in the problem, we find it easier to start with the solution summary and introduce the situation equation as its inverse. Students can unwind (*e*) to get (*f*), and they thereby see a strong connection between the situation equation and the solution summary.

(*e*) $$\frac{125.50 - 64}{3} = \text{price}$$ (the solution summary)

(*f*) $$(3 \times \text{price}) + 64 = 125.50$$ (the situation equation)

Pedagogically, we can show the situation equation as a partner of the solution summary by presenting the two in parallel columns at the whiteboard with the problem statement in the middle. The use of spatial separation and parallel organization helps bridge between the two representations and highlights the complementary roles that each plays in analyzing and solving an algebra story problem.

CONCLUSIONS

Students use a variety of effective solution methods, many of which signal profound understandings of quantitative relationships. Many mathematics teachers are unaware of the power of these invented methods or the role that they can play as a conceptual foundation for formal algebra instruction. The investigation reported here has focused on the simplest of algebraic problems, those depicting a single relationship. In other research, we are learning that as the complexity of the problems increases, as when introducing multiple constraints and negative numbers, problem-solving performance with symbolic representations ultimately surpasses performance that is based only on verbal reasoning (Koedinger and Alibali 1999; Verzoni and Koedinger 1997). Thus, we do not see students' informal problem-solving methods as replacing formal ones. Rather, by building on students' intuitions, teachers can ground important abstract ideas to a conceptual base that supports students' mathematical development. We have found that we can bridge from students' intuitions to formal algebraic representations and solution methods that are part of the long-term goals of mathematics education. Bridging enables students to infuse meaning into abstract symbols and procedures and to see the wide range of mathematical thinking that is within their grasp.

Many teachers are unaware of the power of these invented methods

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