British Journal of Educational Psychology (2008), 00, 1–19

The British Psychological Society

www.bpsjournals.co.uk

Iterating between lessons on concepts and procedures can improve mathematics knowledge

Bethany Rittle-Johnson[1]* and Kenneth Koedinger[2]

[1]Department of Psychology and Human Development, Vanderbilt University, Nashville, Tennessee, USA

[2]Human–Computer Interaction Institute (HCII), School of Computer Science, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA

Background. Knowledge of concepts and procedures seems to develop in an iterative fashion, with increases in one type of knowledge leading to increases in the other type of knowledge. This suggests that iterating between lessons on concepts and procedures may improve learning.

Aims. The purpose of the current study was to evaluate the instructional benefits of an iterative lesson sequence compared to a concepts-before-procedures sequence for students learning decimal place-value concepts and arithmetic procedures.

Samples. In two classroom experiments, sixth-grade students from two schools participated ($N = 77$ and 26).

Method. Students completed six decimal lessons on an intelligent-tutoring systems. In the iterative condition, lessons cycled between concept and procedure lessons. In the concepts-first condition, all concept lessons were presented before introducing the procedure lessons.

Results. In both experiments, students in the iterative condition gained more knowledge of arithmetic procedures, including ability to transfer the procedures to problems with novel features. Knowledge of concepts was fairly comparable across conditions. Finally, pre-test knowledge of one type predicted gains in knowledge of the other type across experiments.

Conclusions. An iterative sequencing of lessons seems to facilitate learning and transfer, particularly of mathematical procedures. The findings support an iterative perspective for the development of knowledge of concepts and procedures.

Children often must learn both fundamental concepts and correct procedures for solving problems in a domain. There is now general consensus that knowledge of concepts and procedures are both important, that they influence one another during

*Correspondence should be addressed to Professor Bethany Rittle-Johnson, Department of Psychology and Human Development, Vanderbilt University, Nashville, TN 37203, USA (e-mail: bethany.rittle-johnson@vanderbilt.edu).

DOI:10.1348/000709908X398106

learning and development, and that early development of concepts is desirable (Baroody, 2003; LeFevre *et al.*, 2006; National Research Council, 2001). However, it is not clear how soon procedures should be introduced to children. Based on an iterative perspective, we hypothesized that early introduction of procedures after an initial concept lesson would benefit learning. In two experiments, we compared sixth graders' learning of decimal place-value concepts and arithmetic procedures after completing a set of computer-based lessons presented in an iterative or concepts-before-procedures sequence.

Relations between knowledge of concepts and procedures

Concepts are ideas generalized from particular instances, especially principles that govern a domain, such as place-value and regrouping concepts for multidigit numbers. In contrast, procedures are step-by-step action sequences to solve problems, such as procedures for adding and subtracting (Blote, Klein, & Beishuizen, 2000; Hiebert & Wearne, 1996; Rittle-Johnson, Siegler, & Alibali, 2001). Debates in psychology and education have often focused on which type of knowledge develops first or is more important (see Baroody, 2003; Hiebert & LeFevre, 1986; Rittle-Johnson & Siegler, 1998; Star, 2005 for reviews). However, these debates may have obscured the gradual development of each type of knowledge and the interactions between the two knowledge types during development. Specifically, knowledge of concepts and procedures may develop iteratively, with increases in one type of knowledge leading to gains in the other type of knowledge, which in turn lead to further increases in the first (Rittle-Johnson, Siegler, & Alibali, 2001).

Past research on mathematics learning is consistent with this iterative perspective. First, children who have greater knowledge of procedures in a domain often have greater knowledge of concepts in that domain (e.g. Byrnes & Wasik, 1991; Cowan & Renton, 1996; Hiebert & Wearne, 1996). Second, longitudinal studies indicate that children's knowledge of concepts and procedures in a domain develop over the same extended period of time (Canobi, Reeve, & Pattison, 2003; Fuson, 1988; LeFevre *et al.*, 2006). Third, improving children's knowledge of procedures can lead to improvements in their knowledge of concepts, and vice versa (Peled & Segalis, 2005; Rittle-Johnson & Alibali, 1999). Finally, prior knowledge of concepts can predict improvements in knowledge of procedures after a problem-solving intervention, which in turn, predicts improvements in knowledge of concepts (Rittle-Johnson *et al.*, 2001). Overall, an iterative perspective is a very promising approach for explaining the development of knowledge of concepts and procedures.

If children typically learn through such an iterative process, modeling instruction after this iterative process could facilitate learning. This perspective implies that concepts and procedures should both be introduced early in the learning cycle, and lessons should iterate between the two. In addition to the following typical developmental process, an iterative sequencing of lessons might support better learning for at least two other reasons. First, gaining some knowledge of procedures reduces demands on working memory during problem solving and may free resources for reflecting on the conceptual underpinnings of the task (Sweller, van Merrienboer, & Paas, 1998). Second, it leads to greater spacing of lessons of a particular type, which has been shown to improve recall (e.g. Cepeda, Pashler, Vul, Wixted, & Rohrer, 2006; Dempster, 1988). In addition, we speculate that it may help highlight the relevance of each lesson type for the other and support knowledge integration.

Most instruction iterates between a focus on concepts or procedures at some level, without exclusive attention to only one type of knowledge over the course of months. However, some theory and instructional practices suggest that knowledge of concepts should be developed extensively before introducing lessons on procedures. In a concepts-before-procedures (*concepts-first*) sequence, knowledge of concepts is developed for an extended period of time before procedures are developed in a particular domain (see Baroody, 2003 for a review). A concepts-first ordering of lessons follows the developmental sequence advocated by prominent theories of cognitive development. Children are said to need knowledge of concepts in order to generate and choose effective procedures (Geary, 1994; Gelman & Williams, 1998; Halford, 1993). Reform efforts in mathematics education in the US advocate for extended development of concepts before introducing procedures. For example, the most widely used reform-oriented middle-school math curriculum in the US, *Connected Mathematics*, dedicates considerable time to developing knowledge of concepts before computational procedures are introduced (e.g. in the sixth-grade series, rational numbers concepts are the focus of unit 4 and rational number procedures are the focus of unit 7; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002).

In support of a concepts-first lesson sequence, some research indicates that prior instruction on procedures can interfere with developing knowledge of concepts (Kamii & Dominick, 1998; Pesek & Kirshner, 2000). For example, Pesek and Kirshner (2000) found that fifth graders who received instruction on procedures before beginning lessons on related concepts performed worse at post-test than students who received the concept, but not the procedure, lessons. Together, research evidence and educational practice suggest that a concepts first sequencing may lead to greater learning than an iterative sequencing because of the potential downsides to practising procedures early in the learning cycle.

Developing knowledge of decimal concepts and procedures

The current study focused on improving children's knowledge of decimal concepts and procedures. Unfortunately, students often have basic misconceptions about decimals (Glasgow, Ragan, Fields, Reys, & Wasman, 2000; Kouba, Carpenter, & Swafford, 1989) and struggle to implement correct procedures for adding and subtracting decimals (e.g. Hiebert & Wearne, 1985).

Our intervention focused on the decimal concepts of place value and regrouping and on procedures for adding and subtracting decimals. In the place-value lessons, students wrote numbers in place-value charts and represented the numbers in several different ways (see Figure 1). In the arithmetic lessons, students read word problems and solved the addition or subtraction problems embedded in the stories (see Figure 2). Each lesson was expected to take students half to one-and-a-half class periods to complete. The task analysis in Table 1 highlights two key features of the lessons. First, the place-value lessons isolated two concepts that students often violate when doing arithmetic – place value and regrouping. Students had the opportunity to explore these ideas without the added demands of identifying relevant quantities in word problems and doing arithmetic computations. Second, the arithmetic lessons could be completed with or without links to the concept lessons. In particular, the alignment step and the borrowing step could be completed using overlapping steps from the place-value task. Thus, this study evaluated whether it is better to fully develop relevant concepts before practising

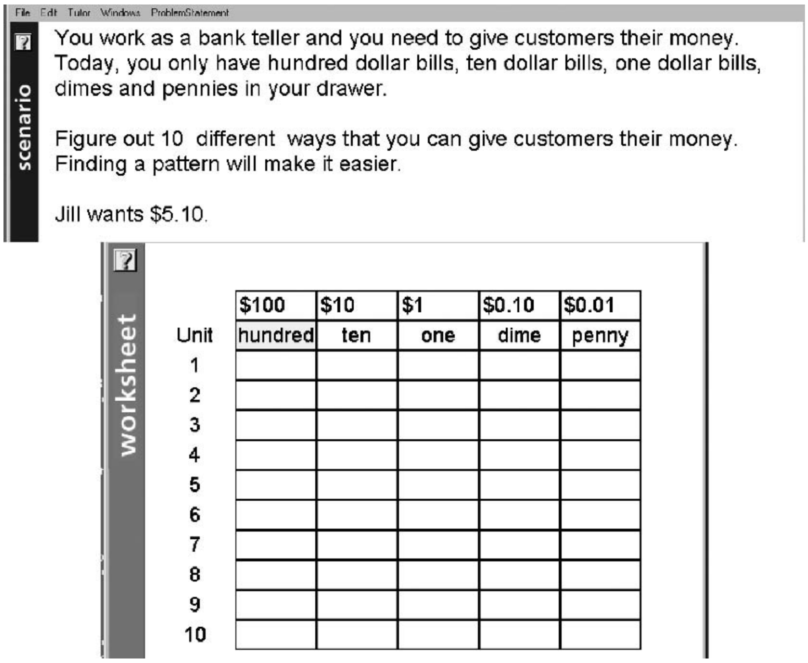

Figure 1. Screen shot of an Experiment 1 place-value item with a money context and unconstrained values.

procedures that can build on those concepts or if it is better to introduce the procedures earlier in the learning cycle. We tested the following hypotheses:

Iterative instruction hypothesis. Based on the iterative model, we hypothesized that an iterative lesson sequence should lead to greater improvements in knowledge of place-value concepts and arithmetic procedures than a concepts-first sequence. Two specific benefits should be to reduce procedural errors and to increase access to regrouping relations.

Iterative relations hypothesis. Knowledge of concepts at pre-test should predict improvements in knowledge of procedures and knowledge of procedures at pre-test should predict improvements in knowledge of concepts.

EXPERIMENT 1

The decimal lessons were embedded in a larger design experiment on creating a sixth-grade math course (Koedinger, 2002). The current study focused on the decimal lessons within the intelligent tutoring system component of the course.

Method

Participants

Initial participants were all 88 sixth graders, who were typically 11 years old, from four mathematics classrooms at two suburban public schools in the US. Of these students, 11 were absent for the pre-test or post-test, leaving 77 students in the final sample, 41 of

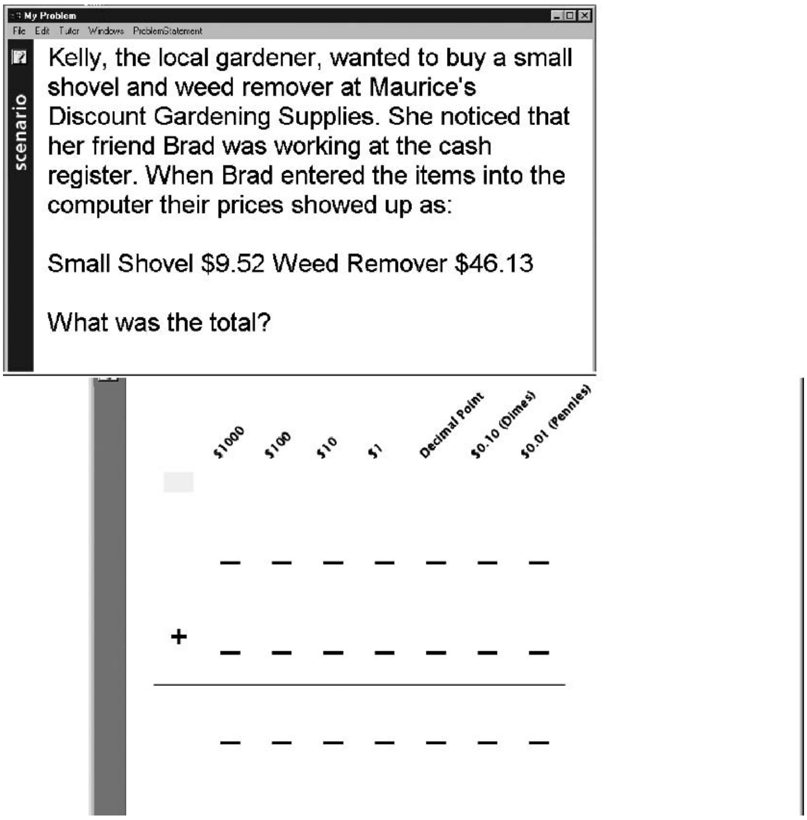

Figure 2. Screen shot of an Experiment 1 & 2 arithmetic item with a money context and place-value labels on the chart.

them female. One mathematics teacher from each school participated, and each teacher taught two sections of mathematics. Students' assignment to class period had not been based on their achievement and teachers followed the same lesson plans with each class.

Intervention

There were three place-value concept lessons and three arithmetic procedure lessons. All children completed the same lesson, which were implemented in our intelligent tutoring system. In the place-value lessons, students were asked to enter a number in a place-value chart and then to show the value of the number in 4–9 novel ways using regrouping (see Figure 1). On the initial items, the method of regrouping was unconstrained. On the later items, one column in the place-value chart was filled in, constraining the possible set of answers and encouraging students to think about the items in different ways. In the first place-value lesson, the items ($N = 16$) were presented in a money context and using money terminology for the place values (as in Figure 1). Using money contexts to introduce decimals is suggested in state standards, in curricula, and by teachers (Glasgow *et al.*, 2000), and familiar contexts can elicit informal strategies and help students avoid nonsensical errors (Koedinger & Nathan, 2004; Rittle-Johnson & Koedinger, 2005). On the two subsequent lessons, the items

Table 1. Task analysis of place-value and arithmetic tasks

Place-value task (see Figure 1 for sample item)	Arithmetic task (see Figure 2 for sample item)
1. Determine place value of left-most digit and type it in the appropriate column. Type remaining digits in appropriate columns to the right	1. Read problem and identify numbers to type in each row of worksheet
2. Pick a digit to reduce and note its place value	2. Align numbers
3. Retrieve regrouping relation: relation between target place value and another place value. (e.g. 1 one is 100 hundredths). (*Borrowing alternative*: always use the relation 1-to-10 for two side-by-side columns, ignoring place value)	a. If column labels: for each number, do step 1 from place-value task
4. Use fact to reduce digit: reduce target digit by retrieved amount	b. If no column labels: type first number. Type second number so the decimal points are in the same column (*Conceptual alternative*: for each number, do place-value step 1 and align digits that overlap in place value)
5. Use fact to increase digit: increase appropriate digit by the retrieved amount.	3. Subtract, beginning with right-most column
6. Type remaining digits, which have kept the same value	a. If top digit is greater than or the same as the bottom digit, then retrieve fact: top digit − bottom digit. Enter fact in answer row. Go to next column and repeat
7. Repeat steps 2–6 as needed	b. If top digit is less than bottom digit, then need to borrow in top row. Reduce the digit in the column to the left by one and add ten to the digit in the current column. Go to next column and repeat. (*Conceptual alternative*: retrieve fact relating current place value to place value of column to the left. Continue with place value steps 4 and 5)

($N = 8$ per lesson) were presented without context and symbolic place-value names were used in the place-value chart (e.g. tenths, hundredths).

In the arithmetic lessons, students were given word problems that required adding or subtracting two decimal numbers. Students entered the numbers in a chart and completed the computations. In the first lesson, items ($N = 7$) were presented in a money context, and monetary place-value column labels were included on the chart (see Figure 2). In the second lesson, items ($N = 8$) were in presented in non-money, and often unfamiliar, contexts and standard place-value labels were included on the chart. Thus, in the first and second lessons, explicit links were made to the place-value lessons. In the third lesson, the items ($N = 30$) were also presented in non-money contexts but the chart did not include place-value labels. Across lessons, the intelligent tutoring system provided customized hint and feedback messages to guide students when they had difficulties.

In the iterative condition, the first place-value lesson was followed by the first arithmetic lesson, followed by the second place-value lesson, and so on. In the

BJEP 642—12/12/2008—ROBINSON—316799

concepts-first condition, all three place-value lessons were presented before the arithmetic lessons.

Assessments

On the pre-test and post-test, students completed six items on decimal place-value and regrouping concepts and six items on decimal arithmetic procedures. Half of the items incorporated a money context and half of the items did not, and items from one version of the assessment are shown in Table 2. The two familiar items for each knowledge type had the same format as items presented during the intervention, but used different numbers. The four novel items for each knowledge type had a different format and required extension of knowledge developed during the intervention. A second version of the assessment was created by switching the numerical values presented in the money and non-money problems, so that money context was not confounded with particular numerical values. The same two forms of the assessment were randomly distributed at pre-test and post-test.

Table 2. Sample assessment items from Experiment 1

	Familiar	Near transfer 1	Near transfer 2
Place-value items			
Money context	Show 10 different ways that you can give Ben \$3.89 (on a place-value chart)	2 dimes are worth how many pennies?	What is one way to show \$7.42 using more than 2 pennies?
No context	List 10 different ways to show the amount 4.07 (on a place-value chart)	3 tenths are worth how many hundredths?	What is one way to show 9.05 using more than 5 hundredths?
Arithmetic items			
Money context	You had \$8.72. Your grandmother gave you \$25 for your birthday. How much money do you have now?	You buy a super-size candy bar for \$1.12, a bag of chips for \$3.39 and a pack of soda for \$4. What is your total cost?	Martha's dinner cost \$8.50. If she gives the waiter \$20.00, how much change should she receive?
No context	Subtract: $64.57 - 8$	Add: $2.29 + 3 + 4.35$	Subtract: 30.00–9.70

Coding

The place-value assessment was worth a possible six points. On the two familiar place-value items, students received a point if at least half of their representations of the number were correct; the other four items had a single answer and were each worth a point. On familiar place-value items, we also scored the number of different, correct regrouping relations a student used to complete the task (see Table 1). A unique regrouping relation was coded each time a student correctly reduced a different digit or combination of digits (e.g. trading a one for 10 tenths, trading 5 ones and 1 tenth for 510 hundredths).

The arithmetic assessment was also worth a possible 6 points, one point for each item answered correctly. Students' incorrect answers were coded for presence of two common errors:

(1) alignment errors, such as aligning numbers on the right, rather than by place value, before computing and
(2) borrowing errors, such as failing to decrement the borrowed-from digit (e.g. $30 - 9.70 = 21.30$).

A second person coded the errors made by half of the participants and the number of regrouping relations used for 20% of participants, and exact agreement was 91% for each. Finally, for each assessment, we calculated students' *gain score* as post-test accuracy − pre-test accuracy.

Procedure

Students participated in a pre-test, intervention and immediate post-test. After completing the pre-test, students worked individually on the lessons during their mathematics period, which lasted 40–50 min a day. Each intervention condition was implemented as a different software application, so class, rather than student, was used as the unit of randomization because it was too confusing for the teachers to ask different students to use different software applications within the same class. For each teacher, we randomly chose one class period to participate in each condition so that condition was not confounded with teacher or school.

Students went to the school computer laboratory twice a week to work on the lessons. They took from four to eight weeks to complete the lessons, with an average time of 4 h, 0 min spent on the lessons ($SD = 75$ min; Range: 114–552 min). Students in the two condition did not differ in the time spent on the intervention, $p = .43$.

As soon as they completed the decimal lessons, students completed the post-test. After eight weeks, nine students had not finished the tutor due to a combination of being absent or not staying on task; six of these students were in the iterative condition. These students were given the post-test without having completed the tutor because the teachers wanted them to move to the next unit. Including these students may underestimate the effect of condition since they did not complete all the intervention material, but we included them to better estimate the effects under normal classroom conditions.

Results and discussion

Pre-test

We confirmed that accuracy was similar for the iterative and concepts-first conditions at pre-test on the arithmetic items and on the place-value items, p's $> .2$ (see Table 3). We also compared accuracy across the assessment on the money context vs. no context items. Students were much more accurate on problems presented in money contexts than without context ($M = 70\%$ correct, $SD = 22$ vs. $M = 39\%$ correct, $SD = 26$), $F(1, 75) = 112.27$, $p < .001$, $\eta^2 = .60$. Money contexts should elicit useful prior knowledge during instruction.

Iterative instruction hypothesis

Students in the iterative condition made greater knowledge gains than students in the concepts-first condition, at least on the arithmetic items (see Figure 3). To confirm the effects of condition on learning, we conducted a mixed-measures ANCOVA on gain

Table 3. Proportion correct on familiar and novel arithmetic and place-value items in Experiments 1 and 2 (standard deviations in parentheses)

	Pre-test		Post-test	
	Concepts-first	Iterative	Concepts-first	Iterative
Experiment 1				
Arithmetic	.64 (.22)	.56 (.32)	.72 (.25)	.79 (.22)
Familiar	.59 (.33)	.46 (.37)	.71 (.32)	.81 (.27)
Novel	.67 (.24)	.62 (.34)	.72 (.28)	.78 (.28)
Place value	.45 (.26)	.51 (.26)	.67 (.26)	.73 (.25)
Familiar	.39 (.35)	.46 (.37)	.72 (.36)	.81 (.30)
Novel	.48 (.28)	.53 (.27)	.64 (.30)	.69 (.30)
Experiment 2				
Arithmetic	.52 (.37)	.23 (.28)	.84 (.19)	.94 (.12)
Familiar	.57 (.39)	.37 (.44)	.96 (.14)	1.0 (.01)
Novel	.47 (.28)	.10 (.19)	.72 (.38)	.87 (.23)
Place value	.23 (.21)	.25 (.35)	.76 (.25)	.61 (.31)
Familiar	.22 (.33)	.27 (.39)	.92 (.19)	.58 (.40)
Novel	.23 (.25)	.22 (.32)	.61 (.41)	.64 (.36)

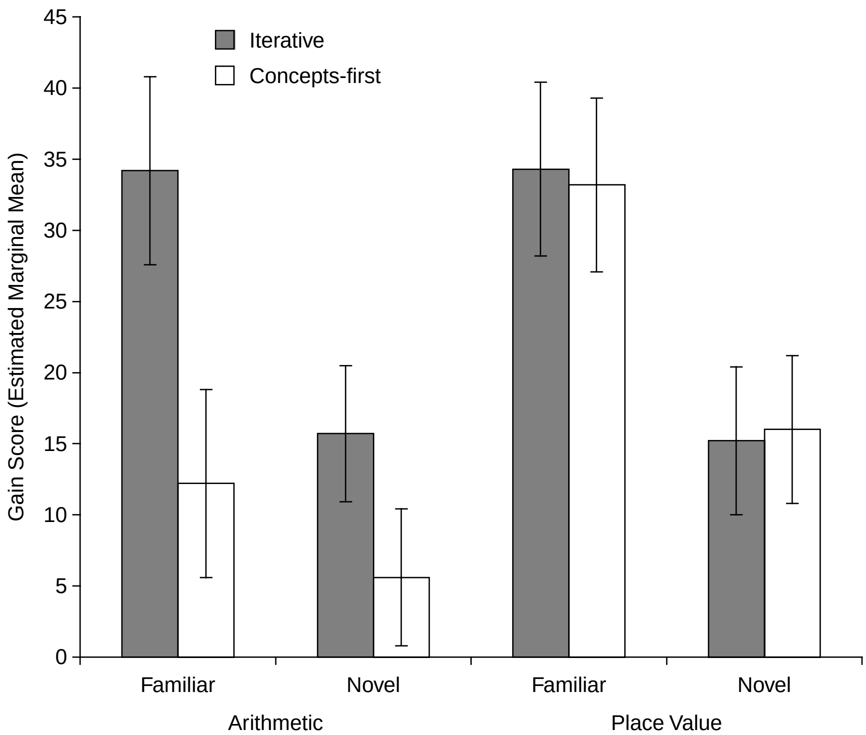

Figure 3. Experiment 1 gain scores by condition for each item type. (Estimated marginal means with standard error bars).

scores, with condition and overall pre-test accuracy as between-subject factors and knowledge type (place value or arithmetic) and familiarity (familiar or novel) as within-subject factors. Preliminary analyses indicated that the effect of condition was similar across the two schools, so a condition by school interaction term was not included in the final model.

Students in the iterative condition made greater knowledge gains than students in the concepts-first condition, $F(1, 74) = 4.55$, $p = .04$, $\eta^2 = .06$ (see Table 3 and Figure 3). To some extent, condition interacted with knowledge type; the effect of condition was somewhat stronger on arithmetic items than on place-value items, $F(1, 74) = 2.97$, $p = .09$, $\eta^2 = .04$. Condition did not interact with familiarity, $p = .38$. The only other significant effect was a main effect for familiarity; gains were greater on familiar items than novel items, $F(1, 74) = 15.85$ $p < .001$, $\eta^2 = .18$. To follow-up the potential interaction between condition and knowledge type, we conducted separate analyses on the arithmetic and place-value assessments. On the arithmetic assessment, students in the iterative condition made greater knowledge gains than students in the concepts-first condition, $F(1, 74) = 7.07$, $p = .01$, $\eta^2 = .09$, and this effect did not interact with familiarity, $p = .29$. On the place-value assessments, students in the two condition made equivalent gains, $p = .98$, and this non-effect did not interact with familiarity, $p = .86$.

One benefit of the iterative sequence was reducing procedural errors. Students in the iterative condition were less likely to make alignment errors on the arithmetic items than students in the concepts-first condition ($M = 8\%$ vs. 20% of the 4 relevant items), $F(1, 72) = 6.04$, $p = .02$, $\eta^2 = .09$, after controlling for differences in accuracy and frequency of alignment errors at pre-test. Borrowing errors were very infrequent at post-test across the two conditions. A second consequence of the iterative sequence was to increase the diversity of regrouping relations used on the familiar place-value items at post-test. Students in the iterative condition used more regrouping relations on the items than students in the concepts-first condition ($M = 1.86$, $SD = 0.69$ vs. $M = 1.54$, $SD = 0.69$), $F(1, 76) = 4.13$, $p < .05$, $\eta^2 = .05$.

Iterative relations hypothesis

We expected prior knowledge of one type to predict improvements in knowledge of the other type. To evaluate this, we regressed pre-test place-value and arithmetic knowledge on gains in arithmetic knowledge and on gains in place-value knowledge. Greater place-value knowledge at pre-test was associated with greater gains in arithmetic knowledge, *partial* $r = .235$, $t(74) = 2.08$, $p = .04$, over and above the effect of pre-test arithmetic knowledge. However, greater arithmetic knowledge at pre-test was not strongly associated with greater gains in place-value knowledge, *partial* $r = .137$, $t(74) = 1.19$, $p = .24$, over and above the effect of pre-test place-value knowledge.

EXPERIMENT 2

Iterating between place-value and arithmetic lessons on decimals facilitated arithmetic learning, compared to completing all the place-value lessons before the arithmetic lessons. In Experiment 2, we sought to replicate this finding using revised lessons and assessments and randomly assigning students to condition.

Method

Participants

Twenty-six sixth-grade students (15 female) agreed to participate. The students were from two classrooms in a small, public, suburban elementary school taught by the same teacher.

Intervention

The place-value lessons were modified so that students only needed to represent a value in three different ways; showing a value in ten ways seemed overly cumbersome for students in Experiment 1. In the arithmetic lessons, items with borrowing across a zero were added and 9 fewer items were presented. Otherwise, the lessons were very similar to the Experiment 1 lessons.

Assessment

The assessment from Experiment 1 was modified to align with the changes to the intervention items and to eliminate money-context items since students were near ceiling on these items at post-test in Experiment 1. For each knowledge type, there were two familiar items and two novel items, and the items are shown in Table 4. Each item was worth one point. We also coded errors, as in Experiment. A second person coded all of the relevant items, and exact agreement was 89–100%.

Table 4. Assessment items from Experiment 2

Item type	Familiar	Novel
Place value	In a place-value chart, enter 5.49 to show the place value of each digit (not scored because students were at ceiling). Then:	a. Which amounts are worth the same amount as 3.5? (need to select at least one correct choice)
	a. Increase the number of hundredths to 19. How many ones and tenths will you need to make 5.49?	b. 2 ones are worth how many hundredths?
	b. What is another new way to show the amount 5.49?	
Arithmetic	a. Add: 64.57 + 29	b. Subtract: 760 − 5.68
	b. Mike and Matt are training for a race. Mike runs 22.4 miles every weekend, and Matt only runs 8 miles. How much further does Mike run than Matt each weekend?	b. Mary is mailing 3 packages. The packages weigh 13 lbs, 0.52 lbs, and 2.5 lbs. What is the total weight of all 3 packages?

Procedure

The procedure was the same as Experiment 1, except that children within a classroom were randomly assigned to condition, resulting in 13 students per condition. The two conditions were implemented in the same software application; we had developed the technical capacity to assign different lesson orders to students using the same application. Students spent an average time of 1 hour and 54 min on the lessons (range was 80–199 min).

Treatment of Missing Data

Five students did not complete the pre-test, and 3 different students did not complete the post-test. Because the proportion of missing data was above 15% at pre-test, statisticians strongly recommend the use of imputation, rather than the traditional procedure of omitting participants with missing data. Simulation studies have found that using Maximum Likelihood (ML) Imputation when data is missing at random leads to the same conclusions as when there is no missing data (Peugh & Enders, 2004; Schafer & Graham, 2002). The data was missing completely at random (confirmed by Little's MCAR test: $\chi^2(16) = 9.20$, $p > .9$). As recommended by Schafer and Graham (2002), we used the Expected Maximization (EM) algorithm for ML estimation via the missing value analysis module of SPSS. The student's missing scores were estimated from all non-missing values that were included in the analyses presented below.

Results and discussion

Pre-test

Although students were randomly assigned to condition, students in the concepts-first condition were more accurate on the arithmetic assessment at pre-test, $F(1, 24) = 4.975$, $p = .035$, $\eta^2 = .17$ (see Table 3). There were no differences between conditions on the place-value assessment ($p > .8$). Pre-test knowledge was included as a covariate in subsequent analyses.

Iterative instruction hypothesis

The effect of condition was not consistent across the different knowledge types (see Figure 4). We conducted a mixed-measures ANCOVA on gain scores, with condition and overall pre-test knowledge as between-subject factors and knowledge type (place value or arithmetic) and familiarity (familiar or novel) as within-subject factors. There was no main effects for condition ($p = .44$), but there were interactions between condition and knowledge type, $F(1, 23) = 21.33$, $p < .001$, $\eta^2 = .48$ and between condition and familiarity, $F(1, 23) = 9.72$, $p = .005$, $\eta^2 = .30$. To better understand these interactions, separate analyses were done on the arithmetic and place-value assessments.

First, consider gains in arithmetic knowledge. As in Experiment 1, students in the iterative condition made greater gains than students in the concepts-first condition, $F(1, 23) = 22.72$, $p < .001$, and the effect was substantial, $\eta^2 = .50$ (see Table 3 and Figure 4). The effect of condition interacted with familiarity such that the effect for condition was larger for novel items than for familiar items (perhaps due to a ceiling effect on the familiar items), $F(1, 22) = 5.25$, $p = .03$, $\eta^2 = .19$. Greater accuracy reflected a reduction in common errors; 25% of students in the concepts-first condition made a borrowing error, whereas none of the students in the iterative condition did. Alignment errors were virtually eliminated across the items at post-test.

Next consider gains in place-value knowledge. The main effect for condition, $F(1, 23) = 4.25$, $p = .05$, $\eta^2 = .16$, was qualified by a condition by familiarity interaction, $F(1, 23) = 10.04$, $p = .004$, $\eta^2 = .30$. As shown in Figure 4, students in

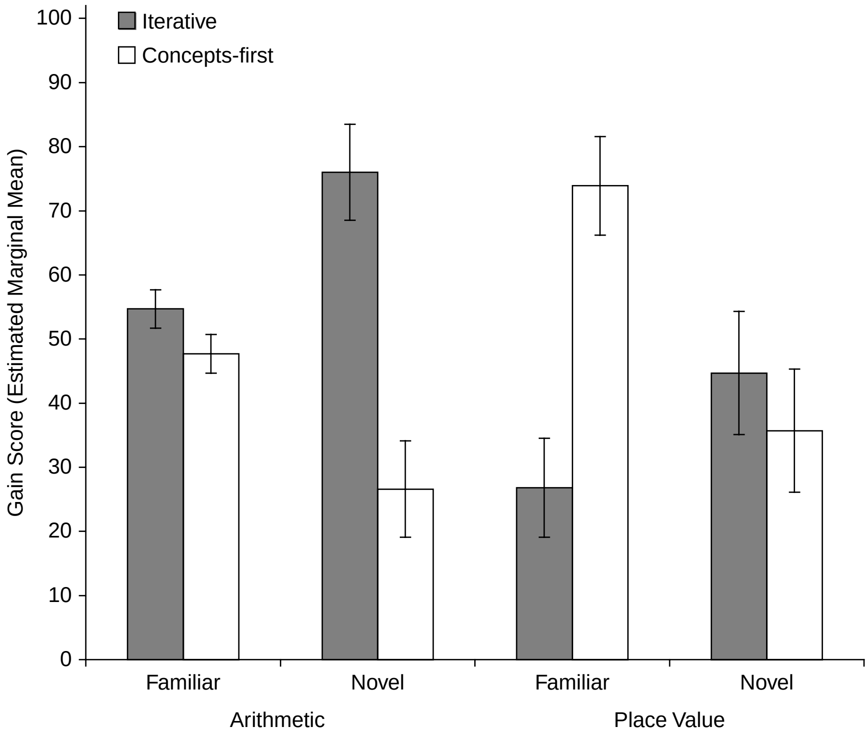

Figure 4. Experiment 2 gain scores by condition for each item type. (Estimated marginal means with standard error bars).

the concepts-first condition made greater gains on the familiar items, but students in the iterative condition made greater gains on the novel items.

Students' responses on the familiar place-value items gave one clue to why students in the concepts-first condition were more accurate on the familiar place-value items, but less accurate on the novel items. Most of the students in the concepts-first condition appeared to use a consistent approach to complete the items (i.e. 5.49 as (a) 5 ones, 3 tenths and 19 hundredths, and (b) 5 ones, 2 tenths and 29 hundredths). This could reflect use of a single regrouping fact or adoption of a borrowing procedure to complete the items. In the concepts-first condition, 75% of students used this approach, whereas only 18% used it in the iterative condition, $\chi^2(1) = 7.43$, $p = .006$. Students in the iterative condition used tenths-to-hundredths regrouping when it was required on the first item, but used other approaches, such as regrouping ones-to-tenths, on the second item. It may be that students in the concepts-first condition relied on a borrowing procedure for solving the items after repeated practice with the format during the intervention. However, this procedure did not seem to transfer to novel problem formats or to reduce borrowing errors on the arithmetic assessment.

Iterative relations hypothesis

As in Experiment 1, greater place-value knowledge at pre-test was associated with greater gains in arithmetic knowledge, *partial* $r = .45$, $t(23) = 2.44$, $p = .02$, over and above the effect of pre-test arithmetic knowledge. In addition, greater arithmetic knowledge at pre-test was associated with greater gains in place-value knowledge, *partial* $r = .51$, $t(23) = 2.81$, $p = .01$, over and above the effect of pre-test place-value knowledge.

GENERAL DISCUSSION

Our hypotheses were generally confirmed, with a few caveats. First, an iterative ordering of lessons generally resulted in larger knowledge gains, particularly for arithmetic knowledge. Second, knowledge of one type predicted knowledge of another type in each of the experiments. We discuss implications of this research for the relations between knowledge of concepts and procedures and for mathematics education.

Implications for the relations between knowledge of concepts and procedures

Knowledge of concepts and procedures may develop best in an iterative process, with improvements in one type of knowledge supporting improvements in the other type of knowledge, supporting further improvements in the first. The current study converges with past research indicating that prior knowledge of one type can influence gains in the other type of knowledge (e.g. Canobi *et al.*, 2003; LeFevre *et al.*, 2006; Rittle-Johnson *et al.*, 2001; Schneider & Stern, 2005). More importantly, it extends prior research by demonstrating that experimentally manipulating the order of instruction to follow an iterative sequence can improve learning, compared to a concepts-before-procedures sequence. This is particularly impressive given that all participants completed the same lessons; only the order of lessons differed.

The iterative lesson sequence consistently improved knowledge of procedures. In particular, students in the iterative condition made fewer arithmetic errors and were better able to transfer their procedures to less familiar arithmetic items, key features of mathematical competence. The iterative sequence seemed to strengthen an arithmetic procedure that utilized concept-supported steps (see Table 1).

The effects of lesson order on knowledge of concepts were less consistent. For the most part, the iterative sequence increased the number of regrouping relations students used, but led to comparable gains in accuracy on the concept items. When the familiar items did not require use of multiple regrouping facts in Experiment 2, the iterative condition led to lower gains on these items. Indeed, three-quarters of students in the concepts-first condition used an approach akin to borrowing, compared to fewer than a quarter of students in the iterative condition. This highlights a potential consequence of presenting all the concept lessons together; students may generate a procedure for completing the items and no longer reflect on the underlying concepts. This may be particularly true on easier tasks.

An iterative perspective moves us beyond debates over which type of knowledge develops first and focuses attention on how an iterative learning cycle might facilitate knowledge change. This study offers clues to at least four potential pathways. First, an iterative cycle could lead to improved knowledge retrieval. Iterating between lessons on concepts and procedures leads to distributed, rather than massed, practice. Distributed practice may lead to greater attention to each new problem, greater variability in the context cues associated with the information, and/or greater retrieval practice, which in turn improves recall of the information (Dempster, 1988; Raaijmakers, 2003; Thios & D'Agostino, 1976). Second, an iterative cycle may improve choices among competing procedures, increasing use of correct procedures and decreasing use of incorrect procedures (Lemaire & Siegler, 1995). Third, iterating between lessons could support knowledge integration. For example, two of the procedural steps could be done with or without links to the relevant concepts (see Table 1). Noticing links between the tasks could support integration of related concepts and procedures. Finally, an iterative cycle may improve adaptation of existing procedures to the demands of novel problems by encouraging use of concepts to evaluate the relevance of known procedures to novel problems and to adapt the known procedure for use on the new problems (Anderson, 1993). Future research is needed to assess the viability of each of these potential pathways.

Educational implications

At least two suggestions for mathematics education can be drawn from the present findings. First, interleaving lessons focused on concepts and procedures may facilitate learning. Fortunately, recent articulation of reform ideas indicates increasing attention to an iterative approach (National Research Council, 2001), and the current study provides critical evidence to support this claim and adds urgency to following the recommendation.

Second, distributing material on a given topic over many weeks, rather than covering it in a single block, may facilitate learning. The benefit of distributed practice (i.e. the spacing effect) is a ubiquitous finding in psychology, but prior research has rarely used school-like activities or been conducted in the classroom (Dempster, 1988). Further, most research on the spacing effect has focused on memorizing facts or completing perceptual-motor skills. The current study provides much needed evidence for extending research on the spacing effect to a moderately complex task and to a classroom setting.

Limitations and future directions

This study raises several additional issues that need to be addressed in future research. First, potential limitations of an iterative sequence for inculcating knowledge of concepts must be addressed. We suspect students in the concepts-first condition made greater gains on the familiar place-value items in Experiment 2 because they learned a narrow procedure for the task. Nevertheless, a potential trade off between the two different lesson sequences for developing knowledge of concepts vs. procedures merits additional research. This research should also include more extensive and varied concept lessons.

Second, an iterative sequence must be compared to a procedures-then-concepts lesson sequence. Some theories of development suggest that children first acquire procedural knowledge and then gain conceptual knowledge from reflecting on the procedures (Inhelder & Piaget, 1980; Siegler, 1991). Students in the current study had some knowledge of procedures at pre-test, and some knowledge of procedures may be necessary to fully benefit from the concept lessons.

Third, the benefits of an iterative sequence must be evaluated across a broad range of mathematical topics and with students who vary in mathematical abilities. Place-value concepts are highly relevant for arithmetic procedures, but in topics in which the concepts and procedures are not as clearly linked, an iterative sequence may be less beneficial. An iterative lesson sequence may also be less beneficial if the target concepts or procedures are more complex. Given individual differences in mathematical abilities (Dowker, 2005), different lesson sequences may also impact students differently. Understanding *when* and *for whom* an iterative lesson sequence benefits learning will inform theories on the development of conceptual and procedural knowledge as well as educational practice.

Fourth, the optimal length and spacing of individual lessons must be explored. An iterative approach suggests breaking lessons into smaller segments than is commonly done in order to interleave lessons on concepts and procedures. But, how much of one type of lesson should precede another type of lesson? Prior research on distributed practice suggests that there will not be a simple rule for determining the best length and spacing of lessons; the optimal spacing of information depends on how long the information needs to be remembered (Cepeda *et al.*, 2006).

Finally, the benefits of an iterative lesson sequence may extend beyond conceptual and procedural lessons. For example, iterating between concrete and symbolic representations of arithmetic procedures within class periods is more effective than presenting lessons using concrete representations before moving on to symbolic representations (e.g. Fuson & Briars, 1990). This suggests that an iterative lesson sequence may have general benefits for knowledge integration that are not specific to conceptual and procedural knowledge. Given the potential benefits of iterative lesson sequences for mathematics learning, these issues merit additional research.

Acknowledgements

This work was supported by NIMH/NRSA training grant 5 T32 MH19983-02 to Carnegie Mellon University and by a grant from Carnegie Learning. Portion of these results were presented at the twenty-fourth Annual Meeting of the North American Chapters of the International Group for the Psychology of Mathematics Education, October, 2002. We greatly appreciate the help of the administrators, teachers and students at Chartiers Valley Middle School and Perrysville Elementary School who participated in the study.

References

Anderson, J. R. (1993). *Rules of the mind*. Hillsdale, NJ: Erlbaum.

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 1–34). Mahwah, NJ: Erlbaum.

Blote, A. W., Klein, A. S., & Beishuizen, M. (2000). Mental computation and conceptual understanding. *Learning and Instruction*, *10*(3), 221–247.

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology*, *27*(5), 777–786.

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (2003). Patterns of knowledge in children's addition. *Developmental Psychology*, *39*(3), 521–534.

Cepeda, N. J., Pashler, H., Vul, E., Wixted, J. T., & Rohrer, D. (2006). Distributed practice in verbal recall tasks: A review and quantitative synthesis. *Psychological Bulletin*, *132*, 354–380.

Cowan, R., & Renton, M. (1996). Do they know what they are doing? Children's use of economical addition strategies and knowledge of commutativity. *Educational Psychology*, *16*(4), 407–420.

Dempster, F. N. (1988). The spacing effect: A case study in the failure to apply the results of psychological research. *American Psychologist*, *43*(8), 627–634.

Dowker, A. (2005). *Individual differences in arithmetic: Implications for psychology, neuroscience and education*. New York, NY: Psychology Press.

Fuson, K. C. (1988). *Children's counting and concept of number*. New York, NY: Springer.

Fuson, K. C., & Briars, D. J. (1990). Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. *Journal for Research in Mathematics Education*, *21*(3), 180–206.

Geary, D. C. (1994). *Children's mathematical development: Research and practical applications*. Washington, DC: American Psychological Association.

Gelman, R., & Williams, E. M. (1998). Enabling constraints for cognitive development and learning: Domain specificity and epigenesis. In D. Kuhn & R. S. Siegler (Eds.), *Handbook of child psychology: Cognition, perception, and language* (5th ed.), (Vol. 2, pp. 575–630). New York, NY: Wiley.

Glasgow, R., Ragan, G., Fields, W. M., Reys, R., & Wasman, D. (2000). The decimal dilemma. *Teaching Children Mathematics*, *7*(2), 89–93.

Halford, G. S. (1993). *Children's understanding: The development of mental models*. Hillsdale, NJ: Erlbaum.

Hiebert, J., & LeFevre, J. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Erlbaum.

Hiebert, J., & Wearne, D. (1985). A model of students' decimal computation procedures. *Cognition and Instruction*, *2*, 175–205.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction*, *14*(3), 251–283.

Inhelder, B., & Piaget, J. (1980). Procedures and structures. In Olson (Ed.), *Social foundations of language and thought* (pp. 19–27).

Kamii, C., & Dominick, A. (1998). The harmful effects of algorithms in grades 1–4. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics. 1998 Yearbook* (pp. 130–140). Reston, VA: National Council of Teachers of Mathematics.

Koedinger, K. R. (2002). Toward evidence for instructional design principles: Examples from Cognitive Tutor Math 6. In D. Mewborn, P. Sztajin, D. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the Twenty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 21–50). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects on representations on quantitative reasoning. *Journal of the Learning Sciences*, *13*(2), 129–164.

Kouba, V. L., Carpenter, T. P., & Swafford, J. O. (1989). Number and operations. In M. M. Lindquist (Ed.), *Results from the fourth mathematics assessment of the national assessment of educational progress* (pp. 64–93). Reston, VA: National Council of Teachers of Mathematics, Inc.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (Eds.), (2002). *Connected mathematics*. Glenview, IL: Prentice Hall.

LeFevre, J. -A., Smith-Chant, B. L., Fast, L., Skwarchuk, S. -L., Sargla, E., Arnup, J. S., *et al.* (2006). What counts as knowing? The Development of conceptual and procedural knowledge of counting from kindergarten through grade 2. *Journal of Experimental Child Psychology*, Q2 *93*(4), 285–303.

Lemaire, P., & Siegler, R. S. (1995). Four aspects of strategic change: Contributions to children's learning of multiplication. *Journal of Experimental Psychology: General*, *124*(1), 83–97.

National Research Council (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Peled, I., & Segalis, B. (2005). It's not too late to conceptualize: Constructing a generalized subtraction schema by abstracting and connecting procedures. *Mathematical Thinking and Learning*, *7*(3), 207–230.

Pesek, D. D., & Kirshner, D. (2000). Interference of instrumental instruction in subsequent relation learning. *Journal for Research in Mathematics Education*, *31*(5), 524–540.

Peugh, J. L., & Enders, C. K. (2004). Missing data in educational research: A review of reporting practices and suggestions for improvement. *Review of Educational Research*, *74*(4), 525–556.

Raaijmakers, J. G. W. (2003). Spacing and repetition effects in human memory: Application of the SAM model. *Cognitive Science*, *27*(3), 431–452.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, *91*(1), 175–189.

Rittle-Johnson, B., & Koedinger, K. (2005). Designing knowledge scaffolds to support mathematical problem solving. *Cognition and Instruction*, *23*(3), 313–349.

Rittle-Johnson, B., & Siegler, R. S. (1998). The relation between conceptual and procedural knowledge in learning mathematics: A review. In C. Donlan (Ed.), *The development of mathematical skills* (pp. 75–110). London: Psychology Press.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346–362.

Schafer, J. L., & Graham, J. W. (2002). Missing data: Our view of the state of the art. *Psychological Methods*, *7*(2), 147–177.

Schneider, M., & Stern, E. (2005). Conceptual and procedural knowledge of a mathematics problem: Their measurement and their causal interrelations. In B. G. Bara, L. Barsalou, & M. Bucciarelli (Eds.), *Proceedings of XXVII Annual Conference of the Cognitive Science Society* (pp. 1955–1961). Mahwah, NJ: Erlbaum.

Siegler, R. S. (1991). In young children's counting, procedures precede principles. *Educational Psychology Review*, *3*(2), 127–135.

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, *36*, 404–411.

Sweller, J., van Merrienboer, J. J. G., & Paas, F. G. W. C. (1998). Cognitive architecture and instructional design. *Educational Psychology Review*, *10*(3), 251–296.

Thios, S. J., & D'Agostino, P. R. (1976). Effects of repetition as a function of study-phase retrieval. *Journal of Verbal Learning and Verbal Behavior*, *15*(5), 529–536.

Received 30 May 2008; revised version received 11 November 2008

BJEP 642—12/12/2008—ROBINSON—316799

Author Queries

JOB NUMBER: 642

JOURNAL: BJEP

Q1 We have truncated the supplied running head as the journal style allows only a maximum of '40' characters. Please approve or provide an alternate.

Q2 Please provide author names for all et al references.