A Cognitive Analysis of the Task Demands of Early Algebra

Hermina J. M. Tabachneck
HCI Institute
School of Computer Science
Carnegie Mellon University
5000 Forbes Avenue
Pittsburgh, PA 15213
tabachneck@cmu.edu

Kenneth R. Koedinger
HCI Institute
School of Computer Science
Carnegie Mellon University
5000 Forbes Avenue
Pittsburgh, PA 15213
koedinger@cmu.edu

Mitchell J. Nathan
Learning Technology Center
Vanderbilt University
Box 45 Peabody
Nashville TN 37203
nathanmj@ctrvax.
vanderbilt.edu

Abstract

Mathematical problems presenting themselves in the workplace and in academia are often solved by informal strategies in addition to or instead of the normative formal strategies typically taught in school. By itself this observation does little to tell us whether, when and how much these techniques should be taught. To ground arguments about the appropriate role of alternative problem-solving techniques in education, we need to first understand the demands of the tasks they address. Our focus here is on algebra and pre-algebra, or, more specifically, on the set of problems that resist solution by more elementary arithmetic methods.

We present a task analysis of this set of problems that is based on the identification of mathematical and situational *problem difficulty factors*. These factors provide a framework for comparing the candidate representations and strategies to meet the demands of more complex problems. We summarize the alternative techniques that have been observed in effective problem solving and discuss their relative strengths and weaknesses. The task analysis along with this comparative analysis provides a basis for hypothesizing developmental sequences and for informing instructional design.

Rethinking Normative Algebra

Human reasoning has often been shown to exhibit certain biases that seem irrational when compared with normative standards (e.g., Kahneman & Tversky, 1973; Wason & Johnson-Laird, 1972). However, reasoning approaches that appear irrational in the context of a narrow set of tasks and norms can turn out to be quite rational when understood within the context of the broader set of task demands to which they are adapted (Anderson, 1990, p. 35 and Polk, 1992 provide such responses respectively to the examples above). For reasoning domains that are taught in school, like mathematics, it is rational, at least at first blush, to take the schooled strategies as normative. However, there is much current activity among educators and cognitive scientists in rethinking school objectives, particularly in mathematics (NCTM, 1989). Many of these efforts have been driven by the empirical observation that people often effectively solve mathematical problems using means other than the school-taught approaches (e.g., Carraher, Carraher, & Schlieman, 1987; Hall, Kibler, Wenger, and Truxaw, 1989; Resnick, 1987). Nevertheless, there are also strong advocates for an emphasis on basic formal skills (e.g., Geary, 1995).

To put this debate on more solid footing, what is needed is an analysis of the task demands for mathematical problem solving and an analysis of the role of both formal and informal approaches in meeting these demands. We have performed a task analysis with the following objectives: 1) to characterize the scope of the task environment and identify the task demands placed on a mathematics problem solver; 2) to present a comparative analysis of the features of the available problem-solving representations and strategies for meeting these demands; and 3) to discuss the implications of this work for specifying a developmental model of mathematical competency that can inform instructional design. This analysis is targeted at algebra and pre-algebra level math.

Scope of the Investigation

The objectives of mathematics instruction beyond arithmetic are largely two-fold: (1) to further develop and refine students' mathematical problem-solving capabilities for everyday life and (2) to prepare students for further studies, particularly in the mathematical sciences. There is an apparent conflict between the symbolic focus of academic math and the recognition that informal non-symbolic methods are used frequently out of school. However, this dichotomy is more apparent than real. Symbolic methods can improve workplace effectiveness and informal methods have always been a part of effective academic science. The recognition that techniques besides symbolic algebra are effective does little to tell us whether, when and how much these techniques should be taught. We need a broader conception of algebra, but also we need a basis for making decisions like these. We need to know what these alternative techniques are good for and what are their limitations.

To start such an investigation, we need to understand what environmental demands these strategies meet. One might characterize these demands as just those tasks that symbolic algebra was designed to address. But using "algebra" in this characterization is circular and overconstrained. Instead, we characterize the scope of the task environment as those tasks for which arithmetic techniques are inadequate or unacceptably inefficient.

What problems are beyond the reach of arithmetic techniques depends both on the achievement level of the

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solver and on the particulars of the problem. We can, however, characterize the difficulty factors that stretch the limits of arithmetic effectiveness and thus, provide likely features for estimating when more advanced methods may be appropriate.

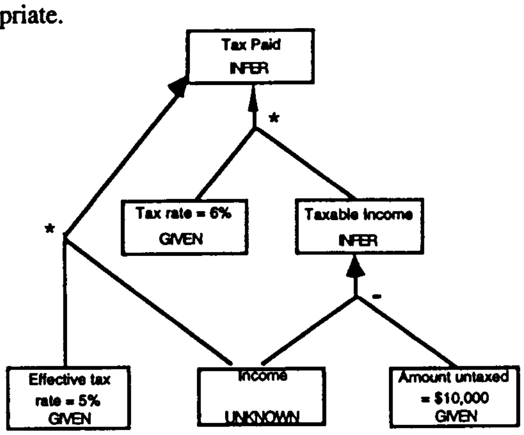

Figure 1: Directed quantitative network for problem P0

Task Demands of Early Algebra

In analyzing the task of mathematical problem solving it is useful to distinguish between "the *quantitative structure* of related mathematical entities and the *situational structure* of related physical entities within a problem" (Hall, et al., 1989, p. 227). Before discussing difficulty factors related to quantitative and situational structure, we begin by presenting a scheme for aiding the analysis.

Directed Quantitative Networks

We use a modified "quantitative network" representation (cf. Hall, et al., 1989; Shalin & Bee, 1985) as an analytic tool for summarizing and clarifying our investigation. Figure 1 shows a *directed* quantitative network for the following problem[1]:

P0. One plan for a state income tax requires those persons with income of $10,000 or less to pay no tax and those persons with income greater than $10,000 to pay a tax of 6 percent only on the part of their income that exceeds $10,000. A person's effective tax rate is defined as the percent of total income that is paid in tax. Based on this definition, could any person's effective tax rate be 5 percent?

The network in Figure 1 shows one way in which the underlying quantities and arithmetic constraints in this problem can be represented. Quantities are represented as nodes and constraints as 3-part directed relations where the quantity at the arrow is the *output* and the other two quantities are *inputs* that are combined with the arithmetic operation to produce the output. For example, the constraint at the top has Tax-rate and Taxable-income as inputs and Tax-paid as the output computed by multiplying the inputs. Some constraints and quantities are explicitly mentioned in a problem, while others (e.g. Taxable income) are implied.

[1]A problem from the 1992 NAEP for which only 3% of US twelve graders provided a satisfactory solution and explanation (NAEP, 1993).

Some Quantitative Factors

Result-Unknown vs. Start-Unknown Problems
Literature on elementary word problem solving shows that problem difficulty is strongly affected by the position of the unknown quantity within the problem statement (e.g., Hiebert, 1982; Staub & Reusser, in press). Problems like P1 in which the unknown quantity is the result of the events being described, tend to be significantly easier than problems like P2 in which the unknown quantity is a start or transition state in the events being described:

P1. Mary had 3 marbles. Then John gave her 5 marbles. How many marbles does Mary have now?

P2. Mary had some marbles. Then John gave her 3 marbles. Now Mary has 5 marbles. How many marbles did Mary have in the beginning?

Riley and Greeno (1988) found that while 1st and 3rd grade students were 100% correct on P1, they were 33% and 95% correct, respectively, on P2. This unknown position effect can be captured within the directed quantitative network. Problems are more difficult when the unknown is an input to a constraint. Problem P0 becomes a result-unknown if income is given.

Mathematical Complexity
Although 3rd graders can solve certain start-unknown problems, like P2, there are numerous other factors that can quickly put start-unknown problems out of reach of elementary students and even many adults. A striking example of this is a local business executive who was struggling with a problem like P3.

P3. 80% of some number is 100. What is the number?

He needed to do a series of calculations where he knew the result (e.g., $100) of taking 80% of some number and wanted to find that number. While the solution procedure of dividing the $100 by 0.8 is analogous to subtracting 3 from 5 in P2, this problem proved daunting. The relevant dimension here is that changing the *type of the numbers* in a problem (e.g., from integers to percents) makes it more difficult to determine the appropriate inversion operation. Had the problem been 15 times some number is 100, he would have had little trouble deciding to divide. Anderson, Reder, & Ritter (in preparation) provide experimental evidence for this difficulty factor.

The *number of arithmetic operators* in the problem is another difficulty factor. Problem P4 is a start-unknown problem where two operations, multiplication and addition, are needed to get to the result.

P4. When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour does Ted make?

A similar one operator problem (e.g., without the tip) is straightforward. However, in a study of urban high school students (Koedinger & Tabachneck, 1995), two operator start-unknown problems like P4 were solved only 59% of the time. They were substantially more difficult than analogous result-unknown problems that were solved 73% of the time. While the Riley and Greeno (1988) study showed the effect of unknown position on one operator problems nearly disappearing by 3rd grade, with the added complexity of two operator problems this effect appears again even for much older students.

Connected vs. Disconnected Problems

Bednarz and Janvier (in preparation) make a distinction between "connected" and "disconnected" problems to shed light on "the passage from arithmetic to algebra". In connected problems "a relationship can be easily established between two quantities thus leading to the possibility of arithmetic reasoning" while disconnected means "no direct links (or bridges) can be directly established between the known quantities" (Bednarz & Janvier, in preparation, p. 10).

P5. Connected problem: Albert has 4 times as many stamps as Judith and 7 times as many as Sophie. If Albert has 504 stamps, how many do the three children have altogether?

P6. Disconnected problem: 380 students are registered in sports activities for the season. Basketball has 76 more students than skating and swimming has 114 more than basket ball. How many students are there in each of the activities?

The difference in difficulty between connected and disconnected problems is large. For example, Bednarz and Janvier report that their middle school age subjects (12 to 13) were 82% correct on the connected problem P5 and only 5% correct on the disconnected problem P6. The disconnectedness dimension has been used as a way of distinguishing between arithmetic and algebra problems (Kieran, 1992, p. 393); however, it seems clear that some connected problems, like P3 and P4 above, require mathematical power at or beyond the edge of arithmetic competency.

Quantitative networks can be used to provide a more precise definition of the connected-disconnected distinction. A problem is connected when a solution can be found by successively propagating the results from constraints with two known values. Problem P0 is disconnected because there are no constraints with two known values. However, if the Tax-rate were unknown and Income given, it would be connected (but not result-unknown).

Some Situational Factors

Situational Facilitation

P7. Situational facilitation: There are 5 birds and 3 worms. How many birds won't get a worm?

P8. Neutral: There are 5 birds and 3 worms. How many more birds are there than worms?

Consider problems P7 and P8. Problem P7 provides situational support for the computation as it suggests a one-to-one matching solution strategy. Nursery school students were 83% correct on P7 but only 17% correct on P8 (Hudson, 1983). These problems are clearly within arithmetic competence; however, they nicely illustrate the kinds of situational facilitation that can occur at any level. Examples that go beyond arithmetic competence will be given below.

Quantitative networks do not capture situational facilitation, except to the extent that a implicit constraint or quantity is more likely to be included in the problem solver's conception of the problem when there is appropriate situational support (Nathan & Resnick, 1993).

Problem Presentation and Context

The mention of "story problems" elicits groans of pain among students and their purported difficulty is ingrained in America culture, so much so that story problems are standard stock for cartoons (e.g., Gary Larson's "Library from Hell" which has only story problem books). Cummins, et al. (1988) comment on this widespread belief: "as students advance to more sophisticated domains, they continue to find word problems in those domains more difficult to solve than problems presented in symbolic format (e.g., algebraic equations)". However, the empirical support for this belief is unclear.

A recent study with ninth graders (Koedinger & Tabachneck, 1995) showed that, all other things being equal, story problems were <u>easier</u> to solve than the analogous algebraic equations. Students were only correct 50% of the time when solving algebraic equations, like P9. They were much better (75% correct) with the addition of a story context, like P10.

P9. (x - 64) / 3 = 26.5

P10. After hearing that Mom won a lottery prize, Bill took the amount she won and subtracted the $64 that Mom kept for herself. Then he divided the remaining money among her 3 sons giving each $26.50. How much did Mom win?

P11. Starting with some number, if I subtract 64 and then divide by 3, I get 26.5. What number did I start with?

Carraher et al. (1987) found a similar effect with 3rd graders on one operator result-unknown problems. They attributed students' relative success on word problems (56%) versus analogous symbolic problems (38%) to effects of "context". Follow-up studies by Baranes, Perry, and Stigler (1989) refined this notion to conclude that it is "relevant context" that counts. However, a second result from Koedinger and Tabachneck (1995) suggests that there may be more going on here. Students did as well on situation-free verbally stated equations like P11 (74% correct) as they did on word problems like P10. The advantages may derive from students' familiarity with words as representations of procedures rather than from any situational content.

Problems that present themselves in natural settings may provide opportunities for situational facilitation that are not present or not as likely in the verbally presented problems in the classroom or laboratory. For example, students' non-sensical answers (e.g., 31 1/3 buses) on test problems like the well-known buses problem (P12) seem less likely if given a real setting (cf. Silver & Shapiro, in press).

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P12. An army bus holds 36 soldiers. If 1,128
soldiers are being bused to their training
site, how many buses are needed?
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Despite much emphasis on situational factors in cognition, there are a number of large gaps in what we know about the characteristics of the everyday/workplace task environment. Our analysis has mostly focused on verbally presented problems in classroom and laboratory contexts. There is no strong evidence at this point to believe that this analysis will not be applicable to naturally presented problems. But, this is an area worth further investigation.

Table 1 summarizes difficulty factors we identified. Problems characterized by easier values of these factors tend to be arithmetic problems. Problems with harder values tend to require competence beyond arithmetic.

Table 1. Difficulty factors that push Arithmetic

Difficulty Factors	**Easier**	. . .	**Harder**
Unknown position	result		start
Connectedness	connected		disconnected
Number of operators	one		many
Number types of quantity	integer	real	complex
Kinds of operators	+ -	*	/ ^
Number-fact facilitation	facilitate	neutral	inhibit
Situational factors	facilitate	neutral	inhibit

Cognitive Representations and Strategies

Although mathematics instruction tends to focus overwhelmingly on symbolic representations and computational procedures of problem solving (e.g., Leinhardt, 1988), students use a variety of alternative methods to address problems. In one study, grade 5 children without training used a trial-and-error strategy exclusively in solving mathematics problems (Lester, 1980). Even matriculated adults with many years of experience using symbolic representations, spontaneously use alternate methods such as guess-and-check or proportional reasoning for solving more complex problems (e.g., Hall et al., 1989; Tabachneck, et al., 1994). There is mounting evidence that students' alternative ways of quantitative reasoning are more complex and efficacious than has been previously suggested in the misconceptions literature (e.g., Smith, diSessa, & Roschelle, 1993).

The fundamental difficulty of problems that push a student to go beyond arithmetic calculations is that it is not possible to produce a solution by propagating given values through the directional constraints implied by the problem. There are two distinguishable classes of problem-solving methods for dealing with this difficulty. The first class of methods is *generate and test:* generate a candidate value for one or more unknowns, propagate it through the constraints, test whether they are met, and if necessary iterate. The second class of methods is *constraint untangling:* reverse the directionality of the constraints and otherwise transform them so that it becomes possible to forward propagate the given values to produce a solution. In constraint untangling, the objects of manipulation are the constraints themselves. In arithmetic and generate-and-test, the constraints are procedures to follow and the quantities they relate are the objects of manipulation. The process of objectifying arithmetic procedures as objects of manipulation evolved over thousands of years in the historical development of algebraic technique and notations (Sfard & Linchevski, 1993). It is perhaps no surprise that this transition is quite difficult for students.

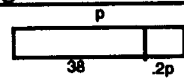
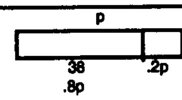
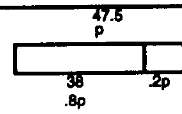
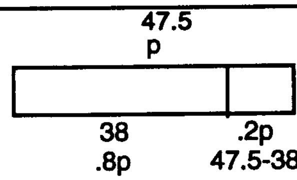
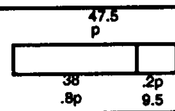
Other things equal these methods are at opposite ends of the "preparation vs. deliberation tradeoff" (Newell, 1990, p. 102). Generate and test methods can be performed with less knowledge (less learning time investment), but tend to be less efficient and require greater deliberation. Constraint untangling methods can be quite efficient, but require significant learning time.

Table 2: Strategies vary by reasoning method and representation.

	Representational formats			
Methods ↓	symbolic	verbal	diagram	situational
Generate And Test	Plug-in	Guess& check	Enumerate	Model-based
Constraint Untangling	Algebra	Verbal algebra	Diagram annotate	Use objects Ratio

As illustrated in Table 2, both classes of methods have instances in a variety of different representational formats. For instance, constraint untangling done in the symbolic format is the traditional algebra strategy. An illustration of this strategy is shown in the first column of Table 3. The next two columns illustrate constraint untangling in two other representational formats which have been observed in studies of verbal and written problem-solving protocols (Hall et al., 1989; Tabachneck et al., 1994). Only the transformations *within* a given representation are illustrated. The difficult process of translating a given problem to one of these solution-enabling representations is addressed below.

Table 3: Instances of the constraint untangling method in three different representational formats on isomorphs of the problem: "I paid $38 for jeans. I got them at a 20% discount. How much did I save?".

Algebra	Verbal algebra	Diagram annotation	Generic operator
p - .2p = 38	The original price minus 20% is $38.		Given
.8p = 38	So, 80% of the original price is $38.		Forward constraint propagation (Simplify)
p = 38/.8 = 47.5	To get the original price, divide $38 by .8 which is 47.5.		Backward constraint propagation (Unwind)
s = p - 38	The amount saved is the original price minus $38.		Given
s = 9.5	That's $9.50.		Constraint combination (Substitute and Simplify)

When the generate and test method is applied to a symbolic representation, the resulting strategy is termed "plug-in" or "plug-and-chug", and is commonly used with algebraic formulae. The verbal method is a commonly invented method used by students across grade levels (e.g., Lester, 1980). The situation-based method reflects an attempt on the part of the solver to model or simulate the events of the problem by, for example, examining earnings at one dollar, then two, etc. These methods, while conceptually simple and easily invented by students of a wide range of achievement levels, are quite powerful, and lie at the core of numerical methods throughout statistics and analytical geometry.

Along the representational format dimension there is a conciseness vs. elaboration tradeoff. Symbolic representations are more concise and abstracted from the problem situation, while situational representations are elaborated with information that is often redundant to, but not essential for the computational process. This redundancy has clear advantages, though. Students who are otherwise quite capable of algebra equation solving (who have paid the preparation price), nevertheless are quite susceptible to difficulties with translation (Hall, et al., 1989).

Compensatory Benefits of Redundant Representations

It stands to reason that different representations and strategies are effective for different kinds of environmental demands. Verbally-based strategies are highly effective at highlighting errors due to mistranslation; diagrammatic strategies, for capturing spatial relations; symbolic strategies, for supporting computation; situational strategies, for simulating processes. In other words, there is no single representation or strategy that is universally effective. Tabachneck et al. (1994) found correlational evidence that the use of multiple strategies during non-routine problem solving yields greater solution success (80%) than the use of a single strategy (40%).

Implications for Instructional Design

The difficulty factors can be used to hypothesize a developmentally appropriate problem sequence. Teachers can use these factors to diagnose a student's zone of proximal development (eg., Brown, 1994) and select problems that are within reach of the student yet present demands that pull her toward more sophisticated mathematical strategies. Introduction of symbolic strategies can provide students with greatly enhanced mathematical power. Nevertheless, teachers should support and encourage students in continued use and reference to their own informal strategies. While often less powerful or general, these informal strategies provide an important source of redundancy that aids students in sense-making, reducing the chance of error and providing a source for self-supervised learning.

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References

Anderson, J.R. (1990) *The adaptive character of thought.* Hillsdale,NJ: Lawrence Erlbaum.

Anderson, J. R., Reder, L. M., & Ritter, S. (in preparation). Algebraic slips: A working memory explanation.

Brown, A. L. (1994). The advancement of learning. *Educational Researcher, 23, 8,* 4-12.

Baranes, R., Perry, M., and Stigler, J. W. (1989). Activation of real-world knowledge in the solution of word problems. *Cognition and Instruction, 6* 287-318.

Bednarz, N. & Janvier, B. (in preparation). Algebra as a problem solving tool: Continuities and discontinuities with arithmetic.

Carraher, T. N., Carraher, D. W., & Schlieman, A. D. (1987). Written and oral mathematics. *Journal for Research in Mathematics Education, 18* 83-97.

Cummins, D. D., Kintsch, W., Reusser, K., & Weimer, R. (1988). The role of understanding in solving word problems. *Cognitive Psychology, 20,* 405-438.

Geary, D. C. (1995). Reflections of evolution and culture in children's cognition: Implications for mathematical development and instruction. *American Psychology, 50,* (1), 24-37.

Hall, R., Kibler, D., Wenger, E., & Truxaw, C. (1989). Exploring the episodic structure of algebra story problem solving. *Cognition and Instruction, 6,* 223-283.

Hiebert, J. (1982). The position of the unknown set and children's solutions of verbal arithmetic problems. *Journal for Research in Mathematics Education. 13,* 5, 341-349.

Hudson, T. (1983). Correspondences and numerical differences between disjoint sets. *Child Development, 54,* 84-90.

Kahneman, D, & Tversky, A.(1973). On the psychology of prediction. *Psychological Review, 80,* 237-251.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of Research in Mathematics Teaching and Learning* (pp. 390-419). New York: MacMillan Publishing Company.

Koedinger, K.R., & Tabachneck, H.J.M. (1995). *Verbal reasoning as a critical component in early algebra.* Paper presented at the 1995 annual meeting of the American Educational Research Association, San Francisco.

Leinhardt, G. (1988). Getting to know: Tracing students' mathematical knowledge from inuition to competence. *Educational Psychologist, 23*(2), 119-144

Lester, F. K. (1980). Research on mathematical problem solving. In R. J. Shumway (Ed.) *Research in Mathematics*. National Council of Teachers of Mathematics.

Nathan, M. J., & Resnick, L. B. (1993) Inference-making during word problem solving. European Association For Research on Learning and Instruction (EARLI), Aug., 1993 (Aix-en-Provence, France).

National Assessment of Educational Progress (1993). Can students do mathematical problem solving? Report # 23-FR01. Washington DC: National Center for Educational Statistics.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: The Council.

Newell, A. (1990). *Unified Theories of Cognition.* Cambridge, MA: Harvard University Press.

Polk, T.A. (1992) Verbal Reasoning. Dissertation, Carnegie Mellon University School of Computer Science #CMU-CS-92-178.

Resnick, L. B. (1987). Learning in school and out. In *Educational Researcher, 16* (9).

Riley, M.S., & Greeno, J.G. (1988). Developmental analysis of understanding language about quantities and solving problems. *Cognition and Instruction, 5,* 49-101.

Shalin, V., & Bee, N.V. (1985). *Analysis of the semantic structure of a domain of word problems.* (Tech. Rep. No. APS-20). Pittsburgh: University of Pittsburgh, Learning Research and Development Center..

Silver, E. A. & Shapiro, L. J. (1992). Examinations of situation-based reasoning and sense-making in students' interpretations of solutions to a mathematics story problem. In J.P.Ponte, J.F. Matos, J.M. Matos & D. Fernandes (Eds.), *Mathematical Problem Solving and New Information Technology: Research in Contexts of Practice.* Springer-Verlag.

Sfard, A. & Linchevski, L. (1993). The gains and pitfalls of reification: The case of algebra. *Educational Studies in Mathematics, 00:* 1-38.

Smith, J. P., III., diSessa, A. A., & Roschelle, J. (1993).Misconceptions reconceived: A constructivist analysis of knowledge in transition. *Journal of The Learning Sciences, 3*(2). 115-164.

Staub, F. C. & Reusser, K. (in press). The role of presentational structures in understanding and solving mathematical word problems. In C.A. Weaver, S. Mannes & R.C. Fletcher (Eds.), Discourse Comprehension: Essays in Honor of Walter Kintsch. Hillsdale, N.J.: Lawrence Erlbaum Associates, in press.

Tabachneck, H. J. M., Koedinger, K. R., & Nathan, M. J. (1994). Toward a theoretical account of strategy use and sense-making in mathematics problem solving. In *Proceedings of the Sixteenth Annual Conference of the Cognitive Science Society.* Hillsdale, NJ: Erlbaum.

Verschaffel, L., De Corte, E. & Lasure, S. (1994). Realistic considerations in mathematical modeling of school arithmetic word problems.

Wason, P.C. & Johnson-Laird, P.N. (1972). *Psychology of reasoning: structure and content.* Cambridge, MA: Harvard Press.